Title	A Trial Production on the Integral III
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Editor(s)	
Citation	Bulletin of University of Osaka Prefecture. Series A, Engineering and natural sciences. 1963, 12(1), p.111-126
Issue Date	1963-09-30
URL	http://hdl.handle.net/10466/8038
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A Trial Production on the Integral III

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(Received June 30, 1963)

In this paper, we study the integration by parts, Stieltjes integral, and the mean value theorems on DC and D_*C integrals. The present definition for D_*C-integral is somewhat wider than the one we have given in II**. If $F(x)$ and $G(x)$ are indefinite DC-integrals of $f(x)$ and $g(x)$ respectively and if $F(x)$ and $G(x)$ are satisfy some conditions, then $(DC)\int_a^b f(x)G(x)dx=F(b)G(b)-F(a)G(a)-(DC)\int_a^b F(x)g(x)dx$. The same proposition with D_*C in the place of DC in the above holds good, too. And, in §6, we give the definition of DC-Stieltjes integral $(DCS)\int_a^b gdF$ and study properties of DC-Stieltjes integral. If $g(x)$ is a function which is DC-integrable and if $F(x)$ is a function of bounded variation, and if g, F satisfy some conditions, then $(DC)\int_a^b g(x)F(x)dx=G(b)F(b)-G(a)F(a)-(DCS)\int_a^b G(x)dF$ where $G(x)$ is an indefinite DC-integral of $g(x)$. In §7, we study the mean value theorems on D_*C and DC integrals. There is a number μ such that $(DCS)\int_a^b gdF=\mu[F(b)-F(a)]$, where μ is a number between the bounds of the function g on $[a, b]$. And, if g is DC-integrable and if F is monotone non-decreasing, and if g, F satisfy some conditions, then $(DC)\int_a^b g(x)F(x)dx$ $=F(a)\cdot(DC)\int_a^\xi gdx+F(b)\cdot(DC)\int_\xi^b gdx$, where ξ is a point of $[a, b]$.

Throughout this paper, all functions are 1-valued, real valued, and real variable functions, and, open sets and closed sets mean open sets and closed sets in the space of all real numbers respectively unless otherwise specified. The closed interval $\{x|a\leq x\leq b\}$ and the open interval $\{x|a<x<b\}$ are represented by $[a, b]$ and (a, b) respectively.

6. Integration by parts

Definition 6.1. Let $f(x)$ be a function defined at almost all points of $[a. b]$, and A be the domain of definition of $f(x)$. If both

$$\lim_{(h\,(a+h)\in A)\to+0}\frac{f(a+h)-f(a)}{h}\quad\text{and}\quad\lim_{(h|(a-h)\in A)\to+0}\frac{f(a)-f(a-h)}{h}$$

exist and if

$$\lim_{(h\,(a+h)\in A)\to+0}\frac{f(a+h)-f(a)}{h}=\lim_{(h\,(a-h)\in A)\to+0}\frac{f(a)-f(a-h)}{h}=\alpha(\neq\pm\infty),$$

tnen $f(x)$ is called derivable at a and we represent it by $f'(a)=\alpha$.

Throughout this paper, '*derivable*' is '*derivable in the sense of Definition* 6.1.'

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** [2].

Theorem 6.1. *If $F(x)$ is a function which is* ACP$_*$ *on* $[a, b]$ *then $F(x)$ is derivable at almost all points of* $[a, b]$.

[*Proof*] Let A and $A_0=\{p_1, p_2, p_3, \cdots\}$ be the domain of definition of $F(x)$ and the non-valued domain of $f(x)$, respectively. For any number $\varepsilon>0$, there is an open set M such that $(a, b)\supset M\supset A_0$ & mM* $<\varepsilon$. Denote M by the union of an enumerable O_k ($k=1, 2, 3, \cdots$) of open intervals, where $O_k\cap O_{k'}=0$ if $k\neq k'$, i.e. $M=\bigcup_k O_k$. Let $O_k=(a_k, b_k)$ ($k=1, 2, 3, \cdots$), and let $\{\varepsilon_j | j=1, 2, 3, \cdots\}$ be a sequence with $\varepsilon_1>\varepsilon_2>\varepsilon_3>\cdots\to 0$.

Now, suppose that $p_1\epsilon O_{k_1}$. If $F(x)$ is bounded on an neighbourhood of p_1, then there is an open interval I_1 satisfying the following conditions (1)—(3):**

(1) $p_1\epsilon I_1\subset \bar{I}_1$*** $\subset O_{k_1}$,

(2) $(\bar{I}_1-I_1)\cap A_0=0$,

(3) There are four points $x_{1,1}$, $x_{1,2}$, $x_{1,3}$, $x_{1,4}\epsilon O_{k_1}-\bar{I}_1$ such that, for all pairs (x, y) $(x\in I_1\cap A,\ -\infty<y<+\infty)$,

$$\tan^{-1}\frac{F(x)-y}{x-a_{k_1}}\leq\tan^{-1}\frac{F(x_{1,1})-y}{x_{1,1}-a_{k_1}}+\varepsilon_1,$$

$$\tan^{-1}\frac{F(x)-y}{x-a_{k_1}}\geq\tan^{-1}\frac{F(x_{1,2})-y}{x_{1,2}-a_{k_1}}-\varepsilon_1,$$

$$\tan^{-1}\frac{y-F(x)}{b_{k_1}-x}\leq\tan^{-1}\frac{y-F(x_{1,3})}{b_{k_1}-x_{1,3}}+\varepsilon_1,$$

$$\tan^{-1}\frac{y-F(x)}{b_{k_1}-x}\geq\tan^{-1}\frac{y-F(x_{1,4})}{b_{k_1}-x_{1,4}}-\varepsilon_1. \quad \left(-\frac{\pi}{2}<\tan^{-1}<+\frac{\pi}{2}\right).$$

And, if $F(x)$ is unbounded on each neighbourhood of p_1, then we shall define that $I_1=O_{k_1}$.

We shall define, by induction, the sequence $\{I_1, I_2, I_3, \cdots\}$ of open intervals and the four sequences $\{x_{j,s}\epsilon I_j | j=1, 2, 3, \cdots\}$ ($s=1, 2, 3, 4$) of points, as follows:

Suppose $p_j\epsilon O_{k_j}$. If $p_j\epsilon \bigcup_{n=1}^{j-1} I_n$ then $I_j=0$. If $p_j\notin \bigcup_{n=1}^{j-1} I_n$ and if $F(x)$ is bounded on an neighbourhood of p_j, then there is an open interval I_j satisfying conditions (4)—(8);

(4) $p_j\epsilon I_j\subset\bar{I}_j\subset O_{k_j}$,

(5) $\bar{I}_j\cap\bar{I}_{j'}=0 \quad \forall j'<j$,

(6) $(\bar{I}_j-I_j)\cap A_0=0$,

(7) $\bar{I}_j\cap\{x_{t,s} | t=1, 2, 3, \cdots, j-1;\ s=1, 2, 3, 4\}=0$,

(8) There are four points $x_{j,1}$, $x_{j,2}$, $x_{j,3}$, $x_{j,4}\epsilon O_{k_j}-\bar{I}_j$ such that, for all pair (x, y) $(x\epsilon I_j\cap A,\ -\infty<y<+\infty)$,

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* mX is the measure of X.

** Choose I_1 sufficiently small and take a point $x_{1,1}$ so that $|F(x_{1,1})-\overline{\lim}_{x\to p_1} F(x)|$ is small enough. Similarly, take points $x_{1,2}$, $x_{1,3}$, $x_{1,4}$. (for sample, $|F(x_{1,3})-\underline{\lim}_{x\to p_1} F(x)|$ is small enough.)

*** $\bar{X}$ is the closure of X.

$$\tan^{-1}\frac{F(x)-y}{x-a_{k_j}}\leq\tan^{-1}\frac{F(x_{j,1})-y}{x_{j,1}-a_{k_j}}+\varepsilon_j\,,$$

$$\tan^{-1}\frac{F(x)-y}{x-a_{k_j}}\geq\tan^{-1}\frac{F(x_{j,2})-y}{x_{j,2}-a_{k_j}}-\varepsilon_j\,,$$

$$\tan^{-1}\frac{y-F(x)}{b_{k_j}-x}\leq\tan^{-1}\frac{y-F(x_{j,3})}{b_{k_j}-x_{j,3}}+\varepsilon_j\,,$$

$$\tan^{-1}\frac{y-F(x)}{b_{k_j}-x}\geq\tan^{-1}\frac{y-F(x_{j,4})}{b_{k_j}-x_{j,4}}-\varepsilon_j\,. \quad \left(-\frac{\pi}{2}<\tan^{-1}<\frac{\pi}{2}\right).$$

And, if $F(x)$ is unbounded on each neighbourhood of p_j, then we shall define that $I_j=O_{k_j}$.*

If $\bar{I}_j\subset O_{k_j}$ for an integer j, then, for all combinations (ξ, x, y) ($\xi\leq a_{k_j}$, $x\epsilon I_j\cap A$, $-\infty<y<+\infty$), we have

$$\Delta_1)\begin{cases}\tan^{-1}\dfrac{F(x)-y}{x-\xi}\leq\tan^{-1}\dfrac{F(x_{j,1})-y}{x_{j,1}-\xi}+\varepsilon_j\,,\\[2ex] \tan^{-1}\dfrac{F(x)-y}{x-\xi}\geq\tan^{-1}\dfrac{F(x_{j,2})-y}{x_{j,2}-\xi}-\varepsilon_j\,,\end{cases}$$

and, for all combinations (ξ, x, y) ($\xi\geq b_{k_j}$, $-\infty<y<+\infty$, $x\epsilon I_j\cap A$), we have

$$\Delta_2)\begin{cases}\tan^{-1}\dfrac{y-F(x)}{\xi-x}\leq\tan^{-1}\dfrac{y-F(x_{j,3})}{\xi-x_{j,3}}+\varepsilon_1\,,\\[2ex] \tan^{-1}\dfrac{y-F(x)}{\xi-x}\geq\tan^{-1}\dfrac{y-F(x_{j,4})}{\xi-x_{j,4}}-\varepsilon_1\,.\end{cases}$$

Set $I=\bigcup_j I_j$, then there is a function Φ which is ACG$_*$ on $[a, b]$ such that

$$\Phi(x)=F(x) \quad x\epsilon([a, b]-I).$$

$\Phi(x)$ is derivable almost everywhere. If Φ is derivable at $x_0\epsilon[a, b]-M$, then, by Δ_1), Δ_2), and $\varepsilon_j\to 0$, $F(x)$ is derivable at x_0. Hence $F(x)$ is derivable at almost all points of $A-M$.

Set $N=\{x\,|\,F(x)$ is derivable at $x\}$, then $mN\geq m([a, b]-M)>(b-a)-\varepsilon$. Thus we have $mN=b-a$. Hence $F(x)$ is derivable almost everywhere on $[a, b]$. *Q.E.D.*

For D$_*$C-integral we give a definition which is more wider than the one which has been given in Definition 4.1'** as follows:

Definition 6.2. A fanction $f(x)$ is termed D$_*$C-integrable on an interval I if there is a function $F(x)$ which is ACP$_*$ on $\bar{I}$ and which has almost everywhere $F'(x)$*** $=f(x)$.

($F(x)$ is called indefinite D$_*$C-integral of $f(x)$ on I.)

Same propositions with 'D$_*$C in the sence of Definition 6.2' in the place of 'D$_*$C' in the §§4—5 hold good, too.

* $\{O_k|F(x)$ is unbounded on $O_k\}$ is a finite set.

** Cf. [2].

*** In the sense of Definition 6.1.

Throughout this paper, 'D_*C-*integral*' is 'D_*C-*integral in the sense of Definition* 6.2.'

Lemma 6.1. *If $F(x)$ and $G(x)$ are two functions which are* ACP* *on* $[a, b]$, *then $F(x)G(x)$ is approximately derivable at almost all points of* $[a, b]$ *and we have*

$$(FG)'_{\text{ap}}(x)=F'_{\text{ap}}(x)G(x)+F(x)G'_{\text{ap}}(x) \quad a.e.$$

[*Proof*] Since both F and G is almost everywhere approximately derivable, hence, for each $x\in[a, b]-P$, where P is a null set, there is a set S_x such that x is a point of density of S_x and both

$$\lim_{(x'|x'\in S_x)\to x}\frac{F(x')-F(x)}{x'-x} \quad \text{and} \quad \lim_{(x'|x'\in S_x)\to x}\frac{G(x')-G(x)}{x'-x}$$

exist.

And, since both $F(x)$ and $G(x)$ is ACP on $[a, b]$, $F(x')\to F(x)$ and $G(x')\to G(x)$ for $x'(\in S_x)\to x$. Hence

$$\lim_{(x'|x'\in S_x)\to x}\frac{F(x')G(x')-F(x)G(x)}{x'-x}=\lim_{(x'|x'\in S_x)\to x}\left(\frac{F(x')-F(x)}{x'-x}G(x')+\frac{G(x')-G(x)}{x'-x}F(x)\right)=F'_{\text{ap}}(x)G(x)+F(x)G'_{\text{ap}}(x).$$

Lemma 6.1′. *If $F(x)$ and $G(x)$ are two functions which are* ACP_* *on* $[a, b]$, *then $F(x)G(x)$ is derivable at almost all points of* $[a, b]$ *and we have*

$$(FG)'(x)=F'(x)G(x)+F(x)G'(x) \quad a.e.$$

Theorem 6.2. *Let $f(x)$ be a function which is* DC-*integrable on* $[a, b]$, *$F(x)$ be an indefinite* DC-*integral of $f(x)$ on* $[a, b]$, *and $G(x)$ be a function which is* ACP *on* $[a, b]$. *If $F(x)G'_{\text{ap}}(x)$ is* DC-*integrable on* $[a, b]$ *and if $F(x)G(x)$ is* ACP *on* $[a, b]$, *then $f(x)G(x)$ is* DC-*integrable on* $[a, b]$ *and we have*

$$(\text{DC})\int_a^b f(x)G(x)dx=F(b)G(b)-F(a)G(a)-(\text{DC})\int_a^b F(x)G'_{\text{ap}}(x)dx.$$

[*Proof*] Since both $F(x)$ and $G(x)$ are ACP on $[a, b]$, by Lemma 6.1,

$$(FG)'_{\text{ap}}(x)=f(x)G(x)+F(x)G'_{\text{ap}}(x) \quad \text{a.e. on } [a, b].$$

Since $F(x)G(x)$ is ACP, $(FG)'_{\text{ap}}(x)$ is DC-integrable on $[a, b]$. Hence $f(x)G(x)=(FG)'_{\text{ap}}(x)-F(x)G'_{\text{ap}}(x)$ is DC-integrable on $[a, b]$ and we have

$$(\text{DC})\int_a^b f(x)G(x)dx=F(b)G(b)-F(a)G(a)-(\text{DC})\int_a^b F(x)G'_{\text{ap}}(x)dx.$$

Corollary. *Let $F(x)$ and $G(x)$ be two functions which are* ACP *and bounded on* $[a, b]$. *If $F(x)G'_{\text{ap}}(x)$ is* DC-*integrable on* $[a, b]$, *then $F'_{\text{ap}}(x)G(x)$ is* DC-*integrable on* $[a, b]$ *and we have*

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* Cf. [1], Definition 3.1.

$$(\mathrm{DC})\int_a^b F'_{\mathrm{ap}}(x)G(x)dx = F(b)G(b) - F(a)G(a) - (\mathrm{DC})\int_a^b F(x)G'_{\mathrm{ap}}(x)dx.$$

Theorem 6.2′. *Let $f(x)$ be a function which is $\mathrm{D}_*\mathrm{C}$-integrable on $[a, b]$, $F(x)$ be an indefinite DC-integral of $f(x)$ on $[a, b]$, and $G(x)$ be a function which is ACP_* on $[a, b]$. If $F(x)G'(x)$ is $\mathrm{D}_*\mathrm{C}$-integrable on $[a. b]$ and if $F(x)G(x)$ is ACP_* on $[a, b]$, then $f(x)G(x)$ is $\mathrm{D}_*\mathrm{C}$-integrable on $[a, b]$ and we have*

$$(\mathrm{D}_*\mathrm{C})\int_a^b f(x)G(x)dx = F(b)G(b) - F(a)G(a) - (\mathrm{D}_*\mathrm{C})\int_a^b F(x)G'(x)dx.$$

Corollary. *Let $F(x)$ and $G(x)$ be two fnuctions which are ACP_* and bounded on $[a, b]$. If $F(x)G'(x)$ is $\mathrm{D}_*\mathrm{C}$-integrable on $[a, b]$, then $F'(x)G(x)$ is $\mathrm{D}_*\mathrm{C}$-integrable on $[a, b]$ and we have*

$$(\mathrm{D}_*\mathrm{C})\int_a^b F'(x)G(x)dx = F(b)G(b) - F(a)G(a) - (\mathrm{D}_*\mathrm{C})\int_a^c F(x)G'(x)dx.$$

Definition 6.3. Let A_0 be a set contained in (a, b) such that the closure $\overline{A_0}$ of A_0 is scattered*. Let $g(x)$ be a function defined on $[a, b]$ and $F(x)$ be a function of bounded variation on $[a, b]$. If a function $H(x)$ defined on $A = [a, b] - A_0$ satisfies the following conditions 5) and 6), then $H(x)$ is called DCS function of $g(x)$ with respect to $F(x)$ on $[a, b]$:

5) For any open set G with $(a, b) \supset G \supset A_0$, there is a function $g_1(x)$ defined on $[a, b]$ such that

$g_1(x)$ is continuouos on $[a, b]$,

$g_1(x) = g(x) \quad (x \in (A - G))$,

$H(x) = (\mathfrak{S})\int_a^x g_1(t)dF(t)^{**} \quad (x \in (A - G))$.

6) For each $p \in (a, b)$,

$$\lim_{(h;(p+h),(p-h)\in A)\to 0} [H(p+h) - H(p-h)] = g(p)\cdot[F(+p) - F(-p)].$$

(A_0 is called non-valued domain of $H(x)$.)

Definition 6.4. Let $g(x)$ be a function defined on $I = [a, b]$ and $F(x)$ be a function of bounded variation on I. If there is a function $H(x)$ which if DCS-function of $g(x)$ with respect to $F(x)$ on I, then $H(b)$ is called DC-Stieltjes integral of $g(x)$ with respect to $F(x)$ over I and is denoted by

$$(\mathrm{DCS})\int_I g(x)dF \quad \text{or} \quad (\mathrm{DCS})\int_a^b g(x)dF.$$

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* Cf. [1], Definition 1.1.

** $(\mathfrak{S})\int_a^b gdF = \int_a^b gdF - (g(a)\cdot[F(a) - F(-a)] + g(b)\cdot[F(+b) - F(b)])$ where $\int_a^b gdF$ is Lebesgue-Stieltjes integral.

Lemma 6.2. *If $H(x)$ is* DCS-*function of $g(x)$ with respect to $F(x)$ on $[a, b]$, then, for each $x \in A$,*

$$(1)\quad [\lim_{(h;(x+h)\in A)\to+0} H(x+h)]-H(x)=g(x)\cdot[F(+x)-F(x)]$$

and

$$(2)\quad H(x)-[\lim_{(h;(x-h)\in A)\to+0} H(x-h)]=g(x)\cdot[F(x)-F(-x)],$$

where A is the domain of definition of $H(x)$.

[*Proof*] Suppose that $p_1>p_2>p_3>\cdots\to p$ & $p_i\in A$ $(i=1, 2, 3, \cdots)$. Then $M=(a, b)-(\{p_i|i=1, 2, 3, \cdots\}\cup p)$ is an open set. And, M contains the non-valued domain of $H(x)$. Hence, by Definition 6.2 and Definition 6.3, there is a continuous function $g_1(x)$ on $[a, b]$ such that

$$\left.\begin{array}{l} g_1(x)=g(x), \\ H(x)=(\mathfrak{S})\int_a^x g_1(t)dF(t) \end{array}\right\} (x\in(A-M)).$$

Since all p_i and p are contained in $A-M$, we have

$$H(p_i)=(\mathfrak{S})\int_a^{p_i} g_1(t)dF(t) \qquad \text{and} \qquad H(p)=(\mathfrak{S})\int_a^{p} g_1(t)dF(t).$$

And, since

$$\lim_{x\to+p}(\mathfrak{S})\int_a^x g_1(t)dF(t)=(\mathfrak{S})\int_a^p g_1(t)dF(t)+g_1(p)\cdot[F(+p)-F(p)],$$

we have

$$\text{iim}_{i\to\infty}(\mathfrak{S})\int_a^{p_i} g_1(t)dF(t)=(\mathfrak{S})\int_a^p g_1(t)dF(t)+g_1(p)\cdot[F(+p)-F(p)].$$

Hence

$$\lim_{x\to\infty} H(p_i)=H(p)+g_1(p)\,[F(+p)-F(p)].$$

Hence we have the relation (1).

We have the relation (2) by a similar argument.

Lemma 6.3. *Let $f(x)$ and $g(x)$ be two functions defined on $[a, b]$ and $F(x)$ be a function of bounded variation on $[a, b]$. If both* DC-*Stieltjes integrals* (DCS)$\int_a^b fdF$, (DCS)$\int_a^b gdF$ *of $f(x)$ and $g(x)$ with respect to $F(x)$ over $[a, b]$ exist, then there is* DC-*Stieltjes integral of $f(x)+g(x)$ with respect to $F(x)$ over $[a, b]$ and we have*

$$(\text{DCS})\int_a^b \{f(x)+g(x)\}dF=(\text{DCS})\int_a^b fdF+(\text{DCS})\int_a^b gdF.$$

[*Proof*] Let $H(x)$ and $\Phi(x)$ be DCS-functions of $f(x)$ and $g(x)$ with respect to $F(x)$ on $[a, b]$ respectively. And let A and B be domains of $H(x)$ and $\Phi(x)$ respectively, and let A_0 and B_0 be non-valued domains of $H(x)$ and $\Phi(x)$ respectively. We shall prove that the function $H(x)+\Phi(x)$ having the domain $A\cap B$ of definition is a DCS-function of $f(x)+g(x)$ with respect to $F(x)$ on $[a, b]$. For any open set G with $(a, b)\supset G\supset A_0$

$\cup B_0$, there are two continuous functions $f_1(x)$, $g_1(x)$ on $[a, b]$ such that

$$\left.\begin{aligned} &H(x)=(\mathfrak{S})\int_a^x f_1(t)dF(t),\\ &f_1(x)=f(x),\\ &\Phi(x)=(\mathfrak{S})\int_a^x g_1(t)dF(t),\\ &g_1(x)=g(x) \end{aligned}\right\} \quad (x\in[a, b]-G).$$

For all $x\in([a, b]-G)$, we have

$$H(x)+\Phi(x)=(\mathfrak{S})\int_a^x f_1(t)dF(t)+(\mathfrak{S})\int_a^x g_1(t)dF(t)=(\mathfrak{S})\int_a^x [f_1(t)+g_1(t)]dF(t),$$
$$f_1(x)+g_1(x)=f(x)+g(x).$$

And, for each $p\in(a, b)$,

$$\begin{aligned} &\lim_{(h\cdot(p-h),(p+h)\in A\cap B)\to 0}\{[H(p+h)+\Phi(p+h)]-[H(p-h)+\Phi(p-h)]\}\\ &=\lim([H(p+h)-H(p-h)]+[\Phi(p+h)-\Phi(p-h)])\\ &=\lim[H(p+h)-H(p-h)]+\lim[\Phi(p+h)-\Phi(p-h)]\\ &=f(p)[F(+p)-F(-p)]+g(p)[F(+p)-F(-p)]\\ &=[f(p)+g(p)][F(+p)-F(-p)]. \end{aligned}$$

Therefore, $f(x)+g(x)$ is DC-Stieltjes integrable with respect to $F(x)$ on $[a, b]$ and we have

$$(\mathrm{DCS})\int_a^b[f(x)+g(x)]dF=(\mathrm{DCS})\int_a^b f(x)dF+(\mathrm{DCS})\int_a^b g(x)dF.$$

Lemma 6.4. *Let $f(x)$ and $g(x)$ be two functions defined on $[a, b]$ and $F(x)$ be a function of bounded variation on $[a, b]$. If both* DC-*Stieltjes integrals* $(\mathrm{DCS})\int_a^b fdF$, $(\mathrm{DCS})\int_a^b gdF$ *of $f(x)$ and $g(x)$ with respect to $F(x)$ over $[a, b]$ exist, then, for any pair $(\alpha;\beta)$ of constants, there is* DC-*Stieltjes integral of $\alpha f(x)+\beta g(x)$ with respect to $F(x)$ over $[a, b]$ and we have*

$$(\mathrm{DCS})\int_a^b[\alpha f(x)+\beta g(x)]dF=\alpha\cdot(\mathrm{DCS})\int_a^b fdF+\beta\cdot(\mathrm{DCS})\int_a^b gdF.$$

[*Proof*] By Lemma 6.3.

Theorem 6.3. *Let $g(x)$ be a function defined on $[a, b]$ and $F(x)$ be a function of bounded variation on $[a, b]$. If there is a* DC-*Stieltjes integral of $g(x)$ with respect to $F(x)$ over $[a, b]$ then the value of the integral is unique.*

[*Proof*] Let $H(x)$ and $\Phi(x)$ be two DCS-fuctions of $g(x)$ with respect to $F(x)$ on $[a, b]$.

Set $\Psi(x)=H(x)-\Phi(x)$. Then, by Lemma 6.3, $\Psi(x)$ is a DCS-function of $\psi(x)=g(x)-g(x)(\equiv 0)$ with respect to $F(x)$ on $[a, b]$. Let A be the domain of definition of $\Psi(x)$ and A_0 be the non-valued domain of $\Psi(x)$.

1°) We shall prove that $\Psi(x)$ is continuous on A.

Suppose that $p\epsilon A$. Then, by Lemma 6.2,

$$[\lim_{(h\ (p+h)\in A)\to+0}\Psi(p+h)]-\Psi(p)=\psi(p)\cdot[F(+p)-F(p)]$$

and
$$\Psi(p)-[\lim_{(h\ (p-h)\in A)\to+0}\Psi(p-h)]=\psi(p)\cdot[F(p)-F(-p)].$$

And, since $\psi(p)=0$, we have

$$\psi(p)\cdot[F(+p)-F(p)]=0$$

and
$$\psi(p)\cdot[F(p)-F(-p)]=0.$$

Hence we have

$$\lim_{(h\ (p+h)\in A)\to+0}\Psi(p+h)=\lim_{(h\ (p-h)\in A)\to+0}\Psi(p-h)=\Psi(p).$$

Hence, $\Psi(x)$ is continuous on A.

2°) Suppose $p\epsilon(A-\overline{A}_0)$. Then there are two numbers c, c' such that $p\epsilon(c, c')$ & $[c, c']\cap\overline{A}_0=0$. Since $\psi(x)\equiv 0$ $(x\epsilon[c, c'])$, by 5) in Definition 6.3, we conclued that $\Psi(x)$ $=$const. on (c, c').

3°) Suppose that $p\epsilon(\overline{A}_0)^{0}$*. Then there exists an open interval (c, c') such that $p\epsilon(c, c')$ & $(c, c')\cap\overline{A}_0=p$. If $c<c_1<c_2<p<c_3<c_4<c'$, then, by 5) in Definition 6.3, $\Psi(x)$ is a constant on each of $[c_1, c_2]$ and $[c_3, c_4]$. Therefore $\Psi(x)=$const. on each of (c, p) and (p, c').

Since $p\epsilon A_0$, by 6) Definition 6.3, we have

$$\lim_{(h\ (p+h),(p-h)\in A)\to 0}[\Psi(p+h)-\Psi(p-h)]=\psi(p)\cdot[F(+p)-F(-p)]=0.$$

Hence $\Psi(x)$ is a constant k_p on $(c, p)\cup(p, c')$.

4°) Suppose that $p\epsilon\overline{A}_0$. By a transfinite induction, we shall prove that there is an open interval (c, c') such that $p\epsilon(c, c')$ & $\Psi(x)=$const. $(\forall x\epsilon A\cap[(c, p)\cup(p, c')])$.

Let α be enumerable ordinal number. Assume that, for each $p\epsilon(\overline{A}_0)^{\gamma}$ $(\gamma<\alpha)$, there is an open interval (c, c') such that $p\epsilon(c, c')$ & $\Psi(x)=$const. on $A\cap[(c, p)\cup(p, c')]$. And we shall prove that, for each $p\epsilon(\overline{A}_0)^{\alpha}$, there is an open interval (c, c') such that $p\epsilon(c, c')$ & $\Psi(x)=$const. $\forall x\epsilon A\cap[(c, p)\cup(p, c')]$.

Suppose that $p\epsilon(\overline{A}_0)^{\alpha}$. Then there is an open interval (c, c') such that $p\epsilon(c, c')$ & $(c, c')\cap(\overline{A}_0-\bigcup_{\gamma<\alpha}(\overline{A}_0)^{\gamma})=p$. If $c<c_1<c_2<p<c_3<c_4<c'$, then each of $[c_1, c_2]$ and $[c_3, c_4]$ is contained in $A\cup(\bigcup_{\gamma<\alpha}(\overline{A}_0)^{\gamma})$. For each $q\epsilon[c_1, c_2]\cup[c_3, c_4]$, by 2) and 3) and the assumption of induction, there is an open interval (c_q, c_q') such that $q\epsilon(c_q, c_q')$ & $\Psi(x)$ $=$const. $(\forall x\epsilon A\cap[(c_q, q)\cup(q, c_q')])$. Each of $[c_1, c_2]$ and $[c_3, c_4]$ is covered by a finite number of these (c_q, c_q') because each of $[c_1, c_2]$ and $[c_3, c_4]$ is compact. Hence $\Psi(x)$ is a constant on each of $A\cap(c, p)$ and $A\cap(p, c')$. Since $\psi(p)=0$, by 6) in Definition 6.2,

* Cf. [1], Definition 1.2.

we have

$$\lim_{(p \mid (p+h),(p-h)\in A)\to 0} [\Psi(p+h)-\Psi(p-h)] = \psi(p)\cdot[F(+p)-F(-p)=0.$$

Hence $\Psi(x)$ is a constant k_p on $A\cap[(c,p)\cup(p,c')]$. And, if $p\in A$, then $\Psi(p)=k_p$ by 1°).

5°) For each $p\in[a,b]$, by 2°)—4°), there is an open interval (c_p, c'_p) such that $p\in(c_p, c'_p)$ & $\Psi(x)=$const. $(\forall x\in A\cap(c_p,c'_p))$. $[a,b]$ is covered by a finite number of these (c_p, c'_p) because $[a,b]$ is compact. Hence $\Psi(x)$ is a constant on $A\cap[a,b]$. As $\Psi(a)=0$, we have $\Psi(x)\equiv 0$ $(\forall x\in A)$, and hence we have $\Psi(b)=0$. Hence we have $H(b)=\Phi(b)$. *Q.E.D.*

By the above theorem, if $g(x)$ is continuous on $[a,b]$, then we have

$$(\mathfrak{S})\int_a^b g\,dF = (\text{DCS})\int_a^b g\,dF.$$

Example.

$$\begin{cases} g(x) = \dfrac{1}{x} & (x\neq 0), \\ g(x)=1 & (x=0). \end{cases}$$

$$\begin{cases} F(x)=x & (x<0), \\ F(x)=x+1 & (x\geq 0). \end{cases}$$

$$(\text{DCS})\int_{-1}^{1} g\,dF = 1.$$

Theorem 6.4. i) *If both* $(\text{DCS})\int_a^c g\,dF$ *and* $(\text{DCS})\int_c^b g\,dF$ $(a<c<b)$ *exist, then*

$$(\text{DCS})\int_a^b g\,dF = (\text{DCS})\int_a^c g\,dF + (\text{DCS})\int_c^b g\,dF.$$

ii) *Let* $g(x)$ *be a function which is* DC-*Stieltjes integrable with respect to* $F(x)$ *on* $[a,b]$. *If either of* $(\text{DCS})\int_a^c g\,dF$ *or* $(\text{DCS})\int_c^b g\,dF$ $(a<c<b)$, *exists then the other also exists and we have*

$$(\text{DCS})\int_a^b g\,dF = (\text{DCS})\int_a^c g\,dF + (\text{DCS})\int_c^b g\,dF.$$

Theorem 6.5. *If* $g(x)$ *is* DC-*Stieltjes integrable with respect to* $F(x)$ *on* $[a,b]$, *then there is a set A, which is at most euumerable, such that*

$$(\text{DCS})\int_a^c g\,dF + (\text{DCS})\int_c^b g\,dF = (\text{DCS})\int_a^b g\,dF$$

for any $c\in[(a,b)-A]$.

Definition 6.5. Let A_0 be a null set contained in (a,b) and $f(x)$ be a function defined on $A=[a,b]-A_0$. If

$$\lim_{(h\ (p+h),(p-h)\in A)\to 0} |f(p+h)-f(p-h)| = 0$$

for a point $p\in[a,b]$, then $f(x)$ is called para continuous at p.

Definition 6.6. Let $g(x)$ be a function defined on $A\subset[a,b]$. We shall say that $g(x)$ is DC-Steltjes integrable on $[a,b]$, if the function $\bar{g}(x)$ which coincides with $g(x)$ on the demain of definition of $g(x)$ and is 0 elsewhere is DC-Stieltjes integrable on $[a, b]$. And, we represent it by

$$(\mathrm{DCS})\int_a^b g dx.$$

Theorem 6.6. *Let $F(x)$ be a function of bounded variation on $[a,b]$ and $g(x)$ be a function which is DC-integrable on $[a,b]$. If an indefinite DC-integral $G(x)$ of $g(x)$ is DC-Stieltjes integrable with respect to $F(x)$ on $[a,b]$ and if $G(x)F(x)$ is para continuous on each non-valued point of $G(x)$, then $g(x)F(x)$ is DC-integrable on $[a,b]$ and*

$$(\mathrm{DC})\int_a^b g(x)F(x)dx=G(b)F(b)-G(a)F(a)-(\mathrm{DCS})\int_a^b G(x)dF.$$

[*Proof*] Let $H(x)$ be a DCS-function of $G(x)$ with respect to $F(x)$ on $[a,b]$, A and B be domains of $G(x)$ and $F(x)$ respectively, and A_0 and B_0 be non-valued domains of $G(x)$ and $F(x)$ respectively.

Set

$$\Phi(x)=G(x)F(x)-G(a)F(a)-H(x).$$

1°) We shall prove that $\Phi(x)$ is continuous on the domain $A\cap B$ of definition.

Let $p_1, p_2, \cdots\to p$ ($p_i\in A\cap B$ $(i=1,2,3,\cdots)$, $p\in A\cap B$). Then $M=(a,b)-[\{p_i\,|\,i=1,2,3,\cdots\}\cup p]$ is an open set containing $A_0\cup B_0$. Hence there is a continuous function $G_1(x)$ on $[a,b]$ such that

$$\left.\begin{aligned}&H(x)=(\mathfrak{S})\int_a^x G_1(t)dF(t),\\&G_1(x)=G(x)\end{aligned}\right\}\ (\forall x\in([a,b]-M)).$$

Represent M by the union $\bigcup_n(a_n,b_n)$ of open sets (a_n,b_n) $(n=1,2,3,\cdots)$ with $(a_k,b_k)\cap(a_{k'},b_{k'})=0\,(k\neq k')$. Then, since $G_1(x)$ is continuous on $[a,b]$, $G_1(x)$ is continuous on $[a_n,b_n]$, hence there is a function $G_{2,n}$ whicn is ACG on $[a_n,b_n]$ such that

$$\inf_{x\in[a_n,b_n]}G_{2,n}(x)\geq\inf_{x\in[a_n,b_n]}G_1(x),\quad \sup_{x\in[a_n,b_n]}G_{2,n}(x)\leq\sup_{x\in[a_n,b_n]}G_1(x),$$

$$G_{2,n}(a_n)=G_1(a_n),\qquad G_{2,n}(b_n)=G_1(b_n),$$

$$(\mathfrak{S})\int_{a_n}^{b_n}G_1(x)dF=(\mathfrak{S})\int_{a_n}^{b_n}G_{2,n}(x)dF.$$

Set

$$\begin{cases}G_2(x)=G(x) & (x\in([a,b]-M)),\\ G_2(x)=G_{2,n}(x) & (x\in(a_n,b_n))\ (n=1,2,\cdots),\end{cases}$$

then $G_2(x)$ is a function which ACG on $[a,b]$ and is $G_2(x)=G(x)$ $(\forall x\in([a,b]-M))$. If $g_2(x)=(G_2)'_{\mathrm{ap}}(x)$ a.e., then

$$\Phi(\xi)=G(\xi)F(\xi)-G(a)F(a)-(\mathrm{DCS})\int_a^\xi G(x)dF=G_2(\xi)F(\xi)-G_2(a)F(a)-$$

$$-(\mathfrak{S})\int_a^\xi G_2(x)dF=(\mathfrak{D})\int_a^\xi g_2(x)F(x)dx \quad (\xi=p_1, p_2, \cdots; p).$$

Since $(\mathfrak{D})\int_a^x g_2(t)F(t)dt$ is continuous on $[a, b]$,

$$(\mathfrak{D})\int_a^{p_i} g_2(x)F(x)dx \to (\mathfrak{D})\int_a^p g_2(x)F(x)dx \quad (i\to\infty).$$

Therefore $\Phi(p_i)\to\Phi(p)$ $(i\to\infty)$, hence $\Phi(x)$ is continuous at p.

2°) If $p\in A_0\cup B_0$, then $\Phi(x)$ is para continuous at p. Because, from the assumption and definitions, $G(x)F(x)-H(x)$ is para continuous at p.

3°) We shall prove that $\Phi(x)$ is a function which is ACP on $[a, b]$.

Let M be an open set containing $A_0\cup B_0$. By an argument similar to the one in 1°), there is a function $G_1(x)$, which is ACG on $[a, b]$, such that

$$\left.\begin{array}{l} H(x)=(\mathfrak{S})\int_a^x G_1(t)dF(t), \\ G_1(x)=G(x) \end{array}\right\} \quad \forall x\in([a, b]-M).$$

If $g_1(x)=(G_1)'_{\mathrm{ap}}(x)$ a.e., then

$$\Phi(x)=G(x)F(x)-G(a)F(a)-(\mathrm{DCS})\int_a^x G(t)dF(t)=G_1(x)F(x)-G(a)F(a)-$$

$$-(\mathfrak{S})\int_a^x G_1(t)dF(t)=(\mathfrak{T})\int_a^x g_1(t)F(t)dt \quad \forall x\in([a, b]-M).$$

Hence we conclued that $\Phi(x)$ is ACP on $[a, b]$.

4°) For any $\varepsilon>0$, there is an open set M such that $M\supset(A_0\cup B_0)$ & $mM<\varepsilon$ & $M\subset(a, b)$. By arguments in 3°), there are two functions $\Phi_1(x)$, $G_1(x)$ which are ACG on $[a, b]$ such that

$$\left.\begin{array}{l} \Phi_1(x)=\Phi(x), \\ \Phi_1(x)=G_1(x)F(x)-G(a)F(a)-(\mathfrak{S})\int_a^x G_1(t)dF, \\ G_1(x)=G(x), \\ (\mathrm{DCS})\int_a^x G(t)dF(t)=(\mathfrak{S})\int_a^x G_1(t)dF(t) \end{array}\right\} \quad \forall x\in([a, b]-M).$$

Almost all points of $[a, b]-M$ are points of density of $[a, b]-M$, and both $G(x)$ and $G_1(x)$ are approximately derivable almost everywhere on $[a, b]$, and $G(x)=G_1(x)$ for almost all points x of $[a, b]-M$. Therefore, since $G'_{\mathrm{ap}}(x)=g(x)$ a.e., $g(x)=(G_1)'_{\mathrm{ap}}(x)$ for almost all points x of $[a, b]-M$.

And, since $\Phi_1(x)=(\mathfrak{D})\int_a^x (G_1)'_{\mathrm{ap}}(x)F(x)dx$, by an argument similar to the above, $\Phi'_{\mathrm{ap}}(x)=(\Phi_1)'_{\mathrm{ap}}(x)=(G_1)'_{\mathrm{ap}}(x)F(x)=g(x)F(x)$ for almost all points x of $[a, b]-M$. Hence

$$m\{x \mid \Phi'_{\mathrm{ap}}(x)=g(x)F(x)\} \geq m([a,b]-M) \geq (b-a)-\varepsilon.$$

Hence $\Phi'_{\mathrm{ap}}(x)=g(x)F(x)$ a.e. on $[a,b]$. Therefore, $g(x)F(x)$ is DC-integrable on $[a,b]$ and we have

$$(\mathrm{DC})\int_a^b g(x)F(x)dx=\Phi(b)-\Phi(a)=G(b)F(b)-G(a)F(a)-(\mathrm{DCS})\int_a^b G(x)dF.$$

7. Mean Value Theorems

Definition 7.1. Let $f(x)$ be a para continuous function on $[a,b]$ with the domain A of definition. If, for any $p \in ([a,b]-A)$, $f(x)$ has no extension which is continuous function defined on $A \cup p$, then $f(x)$ is called fully para continuous on $[a,b]$.

Lemma 7.1. *If $f(x)$ is a para continuous function on $[a,b]$ with the domain A of definition, then there is a function $f_1(x)$ which is fully para continuous on $[a,b]$ such that $f_1(x)=f(x)$ for each $x \in A$.*

[*Proof*] If there is $\mu_p=\lim_{(h;(p-h)\in A)\to+0} f(p-h)$ for a point $p \in ([a,b]-A)$, then we define that $f_1(p)=\mu_p$.

Lemma 7.2. *If $f(x)$ is a fully para continuous function on $[a,b]$, then, for any α, β, μ ($a \leq \alpha < \beta \leq b$: $f(\alpha) \leq \mu \leq f(\beta)$ or $f(\alpha) \geq \mu \geq f(\beta)$), there is a number ξ such that $\mu=f(\xi)$ & $\alpha \leq \xi \leq \beta$.*

[*Proof*] Suppose that $f(\alpha)<\mu<f(\beta)$. Let A and A_0 be the domain and the non-valued domain of $f(x)$ respectively.

Set

$$(A_0)^\nu=\{a_i^\nu \mid i=1,2,3,\cdots\} \quad (\nu=1,2,3,\cdots,\omega,\cdots)$$

(each ν is an enumerable ordinal number, each i is an integer).

We shall define an order (Δ) for a set $M=\{\text{pair } (\nu,i) \mid \nu=1,2,3,\cdots,\omega,\cdots;\ i=1,2,3,\cdots\}$ as follows;

$$(\Delta) \quad (0,1)<(0,2), <\cdots<(1,1)<(1,2)<\cdots<(\omega,1)<(\omega,2)<\cdots.$$

Set

$(\bar{\eta},\bar{i})=\sup^*\{(\eta,i) \mid$ for each $(\nu,j)<(\eta,i)$, there is an open interval $I(a_j^\nu)$ such that there is no pair $(\xi_1;\xi_2)$ of numbers with $f(\xi_1)<\mu<f(\xi_2)$ & $\xi_1,\xi_2 \in I(a_j^\nu)\}$.

If there is such a point $a_{\bar{i}}^{\bar{\eta}}$, then, since $a_{\bar{i}}^{\bar{\eta}}$ is an isolated point of $A_0-\{a_i^\nu \mid (\nu,i)<(\bar{\eta},\bar{i})\}$, there is an open interval I with $a_{\bar{i}}^{\bar{\eta}} \in I \subset (A \cup \{a_i^\nu \mid (\nu,i)<(\bar{\eta},\bar{i})\})$. There are two numbers ξ_1, ξ_2 such that $f(\xi_1)<\mu<f(\xi_2)$ & $\xi_1,\xi_2 \in I$. Since $f(x)$ is fully para continuous, we may assume that $a_{\bar{i}}^{\bar{\eta}}<\xi_1,\xi_2$ or $\xi_1,\xi_2<a_{\bar{i}}^{\bar{\eta}}$. Suppose that $\xi_1<\xi_2<a_{\bar{i}}^{\bar{\eta}}$. Set $\xi=\sup\{c \mid$ for each $x \in [\xi_1,c] \cap A$, $f(x) \leq \mu\}$, theh $f(x)$ is continuous at ξ and $f(\xi)=\mu$. We have same results by similar arguments for other cases $\xi_2<\xi_1<a_{\bar{i}}^{\bar{\eta}}$, $a_{\bar{i}}^{\bar{\eta}}<\xi_1<\xi_2$, $a_{\bar{i}}^{\bar{\eta}}<\xi_2<\xi_1$.

* in the order (Δ).

If there is no point $a_{\bar{i}}^{\bar{\eta}}$ for $(\bar{\eta}, \bar{i})$, we shall set

$$\xi=\sup\{c \mid \text{for each } x\in[\alpha, c]\cap A,\ f(x)\leq\mu\},$$

then, since $f(x)$ is fully para continuous, $f(x)$ is continuous at ξ and $f(\xi)=\mu$.

Theorem 7.1. *Let $g(x)$ be a function which is* DC-*Stieltjes integrable with respect to a monotone non-decreasing function $F(x)$ on an interval $[a, b]$. Then:*

i) $(\mathrm{DCS})\int_a^b g dF=\mu\cdot[F(b)-F(a)]$, *where μ is a number between the bounds of the function $g(x)$ on $[a, b]$.*

ii) *If $S(x)$ is a* DCS-*function of $g(x)$ with respect to $F(x)$ on $[a, b]$, we have $S'(x)=g(x)F'(x)$ at almost all points of $[a, b]$.*

[*Proof*] Let A and A_0 be the domain of definition and non-valued domain of $S(x)$ respectively.

i) Let M be an open set with $(a, b)\supset M\supset A_0$.

There is a function $g_1(x)$, which is continuous on $[a, b]$, such that

$$\left.\begin{array}{l}(\mathrm{DCS})\int_a^x g dF=(\mathfrak{S})\int_a^x g_1 dF,\\ g(x)=g_1(x)\end{array}\right\}\quad \forall x\in(A-M).$$

And, there is a number μ such that

$$(\mathfrak{S})\int_a^b g_1 dF=\mu\cdot[F(b)-F(a)],$$

$$\inf_{x\in[a,b]} g_1(x)\leq\mu\leq\sup_{x\in[a,b]} g_1(x).$$

Suppose that $\mu>\sup_{x\in A} g(x)$. The function $f(x)\equiv\mu$ $(\forall x\in[a, b])$ is DC-Stieltjes integrable and hence $f(x)-g(x)$ is DC-Stieltjes integrable. Since $(\mathrm{DCS})\int_a^b [f(x)-g(x)]dF>0$, we have

$$\mu\cdot[F(b)-F(a)]=(\mathrm{DCS})\int_a^b f dF>(\mathrm{DCS})\int_a^b g dF=(\mathfrak{S})\int_a^b g_1 dF=\mu\cdot[F(b)-F(a)].$$

This is unreasonable. Therefore we have $\mu\leq\sup_{x\in A} g(x)$.

By a similar argument, we have $\mu\geq\inf_{x\in A} g(x)$. And, since $(\mathrm{DCS})\int_a^b g dF=(\mathfrak{S})\int_a^b g_1 dF$, we have

$$(\mathrm{DCS})\int_a^b g dF=\mu\cdot[F(b)-F(a)].$$

ii) For any $\varepsilon>0$, there is an open set M such that $(a, b)\supset M\supset A_0$ & $mM<\varepsilon$. There are two continuous functions $g_1(x)$, $g_2(x)$ such that

$$\left.\begin{array}{l}(\mathrm{DCS})\int_a^b g dF=(\mathfrak{S})\int_a^b g_i dF,\\ g(x)=g_i(x)\end{array}\right\}\quad \forall x\in(A-M)\ \ (i=1, 2),$$

$$(\mathfrak{S})\int_a^x g_1 dF \geq (\mathrm{DCS})\int_a^x g dF \geq (\mathfrak{S})\int_a^x g_2 dF \quad (\forall x \in ((A\cap M) - \bigcup_k I_k))$$

where each I_k is a connected component of M with $O\{(\mathrm{DCS})\int_a^x gdF : I_k\} = \infty$*. Set

$$S_i(x) = (\mathfrak{S})\int_a^x g_i dF \quad (a \leq x \leq b) \quad (i=1, 2),$$

then, $S_1'(x) = S_2'(x) = g(x)F'(x)$ at almost all points of $A-M$. Hence $S(x)$ is derivable at almost all points of $A-M$ and $S'(x) = S_1'(x) = S_2'(x)$ at almost all points of $A-M$.

Set $N = \{x \mid S'(x) = g(x)F'(x)\}$, then $mN \geq m(A-M) > mA - \varepsilon = (b-a) - \varepsilon$. Therefore, we have $S'(x) = g(x)F'(x)$ a.e. on $[a, b]$.

Theorem 7.2. *Let $g(x)$ be a function which is* DC-*integrable on $[a, b]$ and $F(x)$ be a function of bounded variation on $[a, b]$. If a function $G(x)$ which is fully para continuous and is an indefinite* DC-*integral of $g(x)$*** *is* DC-*Stieltjes integrable with respect to $F(x)$ on $[a, b]$ and if $G(x)F(x)$ is para continuous at each non-valued point of $G(x)$, then there exists a point $\xi \in [a, b]$ such that*

$$(\mathrm{DC})\int_a^b g(x)F(x)dx = F(a)\cdot(\mathrm{DC})\int_a^\xi g(x)dx + F(b)\cdot(\mathrm{DC})\int_\xi^b g(x)dx.$$

[*Proof*] Set

$$G_1(x) = G(x) - G(a) \qquad (\forall x \in (\text{the domain of definition of } G(x)),$$

then $G_1(x)$ is an indefinite DC-integral of $g(x)$ and is DC-Stieltjes integrable with respect to F on $[a, b]$, and $F(x)G(x)$ is para continuous at each non-valued point of $G_1(x)$. Hence, by Theorem 6.6, we have

$$(\mathrm{DC})\int_a^b g(x)F(x)dx = G_1(b)F(b) - (\mathrm{DCS})\int_a^b G_1(x)dF(x).$$

From the assumption, $F(x)$ is continuous at each non-valued point of $G(x)$ on $[a, b]$ such that

$$(\mathrm{DCS})\int_a^b G(x)dF(x) = \mu\cdot[F(b) - F(a)].$$

By Lemma 7.2, there exists a number $\xi (a \leq \xi \leq b)$ with $G_1(\xi) = \mu$. Hence we have

$$(\mathrm{DC})\int_a^b g(x)F(x)dx = G_1(b)F(b) - \mu\cdot[F(b) - F(a)] = \mu\cdot F(a) + [G_1(b) - \mu]\cdot F(b)$$

$$= F(a)\cdot G_1(\xi) + F(b)\cdot[G(b) - G(\xi)] = F(a)\cdot(\mathrm{DC})\int_a^\xi g(x)dx + F(b)\cdot(\mathrm{DC})\int_\xi^b g(x)dx.$$

Theorem 7.3. *Given a function $f(x)$ which is* DC-*integrable on $[a, b]$, there exists a number μ such that*

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* $\{I_k\}$ is a finite set. $\left(O\left\{(\mathrm{DCS})\int_a^x gdF : I_k\right\} = \sup_{x\in I_k} (\mathrm{DCS})\int_a^x gdF - \inf_{x\in I_k} (\mathrm{DCS})\int_a^x gdF.\right)$

** If $G_0(x)$ is an indefinite DC integral, then, by Lemma 7.1, there is a function $G(x)$, which is fully para continuous on $[a, b]$, such that $G(x) = G_0(x)$ $(\forall x \in (\text{the domain of definition of } G_0(x)))$. $G(x)$ is an indefinite DC-integral of $g(x)$ on $[a, b]$, too. Therefore, if $g(x)$ is DC-integrable, there exists a function which is an indefinite DC-integral of $g(x)$ and is fully para continuous.

$$(\mathrm{DC})\int_a^b f(x)dx=\mu(b-a),$$

$$\lim_{x\in A} f(x)\leq\mu\leq\sup_{x\in A} f(x)$$

where A is the domain of definition of $f(x)$.

Particularly, if $f(x)$ is fully para continuous on $[a, b]$, there exists a number ξ $(a<\xi<b)$ with

$$(\mathrm{DC})\int_a^b f(x)dx=f(\xi)\cdot(b-a).$$

The same proposition with $\mathrm{D}_*\mathrm{C}$ *in place of* DC *holds good, too.*

[*Proof*] Set

$$(1)\qquad \mu=\frac{(\mathrm{DC})\int_a^b f(x)dx}{b-a}.$$

Suppose that $\sup_{x\in A} f(x)<\mu$. Then

$$(\mathrm{DC})\int_a^b [\mu-f(x)]\,dx=\mu(b-a)-(\mathrm{DC})\int_a^b f(x)dx>0\,.$$

This is contradictory to (1). Hence we have $\sup_{x\in A} f(x)\geq\mu$.

By a similar argument, we have $\inf_{x\in A} f(x)\leq\mu$.

Particularly, in the case that $f(x)$ is fully para continuous: If $f(x)\equiv\mu$ $(\forall x\in A)$, there is a number ξ $(a<\xi<b)$ with $f(\xi)=\mu$. If $\sup_{x\in A} f(x)>\inf_{x\in A} f(x)$, then, from definitions of $f(x)$ and μ, $\sup_{x\in A} f(x)>\mu>\inf_{x\in A} f(x)$. Hence there are two numbers ξ_1, ξ_2 $(a\leq\xi_1,$ $\xi_2\leq b)$ with $\sup_{x\in A} f(x)>f(\xi_1)>\mu>f(\xi_2)>\inf_{x\in A} f(x)$. By Lemma 7.2, there is a number ξ such that $\xi_1<\xi<\xi_2$ (or $\xi_1>\xi>\xi_2$) & $f(\xi)=\mu$. This number ξ has $a<\xi<b$.

Constructive definitions of DC and $\mathrm{D}_*\mathrm{C}$-integrals shall be studied in Part IV. DC (*or* $\mathrm{D}_*\mathrm{C}$)-integral fulfils the Harnack's condition H (*or* H_*), and fulfils a modified condition C of the Cauchy's condition. And, as is known, if there is the principal value of the integral, then the value is eqeal to the value of DC (*or* $\mathrm{D}_*\mathrm{C}$)-integral. The condition C contains the condition on the principal value of the integral. Finally, DC (*or* $\mathrm{D}_*\mathrm{C}$)-integral possible to be defined by generalizations defined by the conditions C, H (or H_*) and a transfinite induction starting with $\mathfrak{L}$-integral.

References

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2) Y. Hayashi: A trial production on the integral II, Bull. Univ. Osrka Pref. Ser. A, **11** No. 2 (1963), 117—126.

Correction to 'A trial production on the integral I'. Page 122, line 32. For '$\bigcup_{\alpha<\beta} X^{\beta}$' should read '$\bigcup_{\beta<\alpha} X^{\beta}$'.

Correction to 'II'. Page 118, line 18. For 'set $\Phi(x)=F(x)+G(x)$, then' should read 'set $\Phi(x)=F(x)+G(x)$. If the closure of the non-valued domain of each of $F(x)$ and $G(x)$ is a null set, then'.