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Doctoral Thesis at Osaka Prefecture University

RELIABILITY ANALYSIS AND RELIABILITY-BASED OPTIMIZATION OF COMPOSITE LAMINATED PLATE SUBJECT TO BUCKLING

NOZOMU KOGISO

February 1997
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February, 1997

Nozomu KOGISO
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Notation

Alphabetic Symbols

$A$  extension stiffness matrix of laminated plate. $A = A_{ij}$, $i, j = 1, 2, 6$.
$A^*$ reduced extension stiffness.
$a$ plate length.
$B$  coupling stiffness matrix of laminated plate, $B = B_{ij}$, $i, j = 1, 2, 6$.
$B^*$ reduced coupling stiffness.
$b$ plate width.
$C_{ij}$ cross check index for the mode tracking.
$C_X$ covariance matrix of random variables, $X$.
$C_Y$ covariance matrix of random variables, $Y$.
$\text{Cov}(\cdot)$ coefficient of variation of variable, $\cdot$.
$\text{Cov}[-, -]$ covariance between two variables, $\cdot$ and $\cdot$.
$\overline{\text{Cov}}$ estimated coefficient of variations by Monte Carlo simulation.
$D$  flexural stiffness matrix of laminated plate, $D = D_{ij}$, $i, j = 1, 2, 6$.
$D^*$ reduced bending stiffness.
$E[\cdot]$ mean value of $\cdot$.
$E_1$ Young’s modulus in the fiber direction.
$E_2$ Young’s modulus in the lateral direction.
$E_6$ shear modulus.
$f_X(X)$ joint probability density function.
$g(\cdot)$ limit state function in $U$-space.
$h$ plate thickness.
$h_0$ ply thickness.
$I$ unit matrix.
$K$  stiffness matrix in buckling analysis.
$K_G$ geometry stiffness matrix in buckling analysis.
$K_{mn}$ diagonal element of the stiffness matrix $K$.
$K_{ij}$ off-diagonal element of the stiffness matrix $K$.
$K_{Gmn}$ diagonal element of the stiffness matrix $K_G$.
$K_{nn}^{ij}$  off-diagonal element of the stiffness matrix $K_G$.  

$M$  moment resultant, $M = (M_x, M_y, M_{xy})^T$.  

$M$  the number of assumed series of out-of-plane displacement function in $y$-direction.  

$N$  in-plane stress resultant or an applied load, $N = (N_x, N_y, N_{xy})^T$.  

$N^*$  standardized applied load.  

$N$  the number of assumed series of out-of-plane displacement function in $x$-direction.  

$N_f$  the number of failure modes.  

$n$  the number of plies.  

$P_e$  element failure probability.  

$P_f$  system failure probability.  

$\hat{P}_f$  estimated failure probability obtained by Monte Carlo simulation.  

$P_i$  failure probability of the $i$-th failure mode.  

$P_{ij}$  joint failure probability of $i$-th and $j$-th failure modes.  

$P_L$  Ditlevsen's lower limit of the system failure probability.  

$P_{UL}$  Ditlevsen's upper limit of the system failure probability.  

$Q_{ij}$  ply stiffness with respect to the plate axis, $i, j = 1, 2, 6$.  

$Q_{ij}$  ply stiffness of the material principal axis, $i, j = 1, 2, 6$.  

$R$  plate aspect ratio, $R = a/b$.  

$R$  Reliability.  

$SD(*)$  standard deviation of variable *.  

$S_x, S_y$  plate edges along $x =$constant or $y =$constant, respectively.  

$t$  lamination sequence variable:  

$U$  strain energy.  

$U, u$  standard normally distributed random vector.  

$U_i$  material invariants, $i = 1, \cdots, 5$.  

$u^*$  design point.  

$w$  out-of-plane displacement.  

$x$  coordinate in plate length.  

$X, x$  random variabels.  

$y$  coordinate in plate width.  

$Y, y$  transformed independent random variables.  

$z$  coordinate in plate thickness.
Greek Symbols

\(\alpha\)  
normalized vector of negative direction cosine of \(u^*\).

\(\beta\)  
reliability index.

\(\beta^e\)  
element reliability index.

\(\beta^e_{\text{even}}\)  
reliability index of \(i\)-th eigen mode \(m + n = \text{even}\).

\(\beta^o_{\text{odd}}\)  
reliability index of \(i\)-th eigen mode \(m + n = \text{odd}\).

\(\beta_L\)  
generalized reliability index of Ditlevsen’s lower bound.

\(\beta^*\)  
generalized reliability index of the system.

\(\beta_{\text{sym}}\)  
generalized reliability index of symmetric laminate model.

\(\beta_{\text{un}}\)  
generalized reliability index of unsymmetric laminate model.

\(\varepsilon^0\)  
mid-plane strains, \(\varepsilon = (\varepsilon_x, \varepsilon_y, \varepsilon_{xy})^T\).

\(\theta\)  
ply orientation angle.

\(\kappa\)  
mid-plane curvature, \(\kappa = (\kappa_x, \kappa_y, \kappa_{xy})^T\).

\(\lambda_i\)  
i-th eigenvalue.

\(\lambda^i_{\text{even}}\)  
i-th eigenvalue obtained by the buckling analysis belonging to \(m + n = \text{even}\).

\(\lambda_{\text{min}}\)  
buckling load factor which is minimum eigenvalue.

\(\lambda^i_{\text{odd}}\)  
i-th eigenvalue obtained by the buckling analysis belonging to \(m + n = \text{odd}\).

\(\mu\)  
inverse of the buckling load, \(\mu = 1/\lambda\).

\(\nu\)  
Poisson’s ration.

\(\mu_{X_i}\)  
mean of random variable \(X_i\).

\(\rho\)  
correlation coefficient matrix between failure modes.

\(\rho_{X_i,X_j}\)  
correlation coefficient between random variables \(X_i\) and \(X_j\).

\(\sigma_{X_i}\)  
standard deviation of random variable \(X_i\).

\(\Phi_m(\bullet)\)  
m-dimensional normal distribution function.

\(\phi_m(\bullet)\)  
m-dimensional normal density function.

\(\phi_i\)  
eigenvector corresponding to \(i\)-th eigenvalue, \(\lambda_i\).

\(\phi_{mn}\)  
coefficient of the assumed series of out-of-plane displacement.

Others

\(\bullet\)  
mean value of \(\bullet\).
Chapter 1

INTRODUCTION

1.1 Abstract

Laminated composite plates are widely used in structural applications because of their high specific strengths (failure stress/unit weight) and specific stiffnesses (stiffness/unit weight). As in the case of any plate, the presence of in-plane loads may cause buckling, Leissa (1987). Therefore, an exact evaluation of the buckling load is important to design a composite structure.

The design problem of a composite plate is to maximize the buckling load or to minimize the weight subject to the buckling load constraint in terms of the stacking sequence. The problem can be formulated as the structural optimization problem. A lot of studies have been conducted on the optimum design problems of composite plates, e.g., Haftka and Gürdal (1992).

However, most of them yield the optimal fiber orientation angles under deterministic conditions, where their material properties and the load conditions are assumed to have no variations. It has been known that such a deterministic optimal design is strongly anisotropic and sensitive to the change in loading conditions, Haftka and Gürdal (1992) and Park (1992). Therefore, it is necessary to consider the effect of such variations in loadings and material properties by applying the structural reliability theory, Thoft-Christensen and Murotsu (1986). Especially, the reliability-based design which maximizes the structural reliability is important, Thoft-Christensen and Murotsu (1986), Sørensen (1986) and Enevoldsen (1991).

The reliability analysis and the reliability-based design under the in-plane strength by the first ply failure criterion have already been studied, Miki et al. (1990), (1992), Shao et al. (1993) and Murotsu et al. (1994). The studies have shown that the reliability attains higher values as the number of fiber axes is increased, and that the reliability-based design approaches a quasi-isotropic laminate construction. The design is very different from the deterministic optimum design in which the orientation angle runs along a loading direction. However, the effect of the random variations of such design parameters of the composite
laminated plate on the buckling load has not been investigated yet.

The motivation of this thesis is to apply the structural reliability theory and the reliability-based optimization to a laminated composite plate subject to buckling. By modeling the buckling failure of the laminated plate and formulating the computation algorithms, the effect of the variations in design parameters such as material constants, ply orientation angles and applied loads are investigated on reliability and optimum design.

1.2 Reviews

Reliability-based optimization is to find an optimal structural design due to the reliability requirements, which requires both structural optimization techniques and reliability methods. Therefore, developments of reliability-based optimization is strongly related to these two fields. Also, studies of optimum design of a composite laminated material are related to the thesis. In this section, the above researches are summarized.

Optimal Design of Composite Laminated Plate

Structural optimization techniques have been developed for the purpose of saving materials or achieving the design requirements. Through the developments of both the computation hardware and the computer algorithms, the structural optimization has been widely spread in many practical applications.

The optimization problem can be divided into two types according to properties of the design variable; continuous and discontinuous problems. In practical applications, the structural elements are often selected from the prescribed dimensions legislated by standards. Therefore, the problem can be formulated as a discontinuous design problem such as integer programming problem or some stochastic method like genetic algorithm or simulated annealing.

However, in order to avoid the difficulty due to the discontinuity, the optimization problems have been solved by using continuous variables, which can be formulated as a constrained nonlinear programming problem. Nowadays, the sequential quadratic programming (SQP) algorithm is known to be one of the most powerful tool, Ibaraki and Fukushima (1991). The detail of the algorithms is described in Appendix A.

In applications which require light weight as well as high performance, laminated composite plates have been widely used because of their high specific strengths and specific stiffnesses. In order to achieve the requirements, a lot of studies are conducted on the optimum design of laminated composites.

Laminated composite plates are composed of layers of orthotropic material with arbitrary orientation angles. The stiffness of the laminated plate can be manipulated by changing the number of layers and their orientations. Therefore, use of these quantities as
design variables enables us to change the material properties of a laminate as well as its thickness.

Optimizations of laminated composite plates subject to buckling are mainly studied under the assumption of orthotropic plates subject to biaxial compression load under simply supported conditions. For continuous variable problems, ply thicknesses, e.g., Schmit and Farshi (1977), and ply orientation angles, e.g., Pedersen (1987), are treated as design variables. Miki (1986) introduced lamination parameters and showed that the buckling load maximization design is achieved by an angle-ply construction with one orientation angle $[\pm \theta]_s$ under the orthotropic assumption.

In realistic designs, the orientation angles are generally limited to the $0^\circ$, $\pm 45^\circ$ and $90^\circ$ or every $15^\circ$ between $0^\circ$ and $90^\circ$, and plate thicknesses are integer multiples of ply thickness. This means that the design problem is to determine the stacking sequence which is formulated by using discrete variables. An integer programming problem, e.g., Haftka and Walsh (1992), and stochastic methods such as simulated annealing, Lombardi (1990) and genetic algorithms, Le Riche and Haftka (1993), are applied to solve the stacking sequence problem.

The optimization studies are expanded to the stiffener design problem whose buckling load is evaluated by numerical calculation. Incorporating the numerical analysis into the optimization code, computation systems such as PASCO, Stroud and Anderson (1981) and Anderson et al. (1981), and PANDA2, Bushnell (1987), are developed. For more general structural design problems which require finite element method as a structural analysis tool, two level optimization with nonlinear programming and genetic algorithms is studied by Yamazaki and Sengokudani (1996). As the first level, the optimum lamination parameters are determined by sequential linear programming. Then, the stacking sequence is determined by genetic algorithm as the second level.

However, most of them yield the optimal fiber orientation angles under deterministic conditions, where the material properties and the load conditions are assumed to have no variations. It has been known that such a deterministic optimal design is strongly anisotropic and sensitive to the change in loading conditions, Haftka and Gürdal (1992) and Park (1992).

Reliability Analysis

There exist uncertainties in loads, strengths and modeling. In order to account for various uncertainties and to satisfy design requirements, safety factors are used extensively without truly understanding the impact of the uncertainties. The approach results in either underestimation or overestimation. Overestimating results in a conservative but costly design and underestimating results in an unsafe or even deadly design.

From the probabilistic point of view, structural reliability theory has been developed,
e.g., Thoft-Christensen and Baker (1986), and Madsen et al. (1986). In the structural reliability theory, these uncertainties are treated as a random vector and a structural failure is modeled as a function of the random vector which is called a limit state function. The structural reliability can be estimated by the first order reliability method (FORM) for a given limit state function. The FORM is a method where the random vector is transformed to the standardized normal distribution space and then failure probability $P_f$ is obtained by taking Taylor's series expansion of the limit state function at the most probable point which needs to be found by iteration or optimization method. Reliability index is defined as $eta = -\Phi^{-1}(P_f)$, where $\Phi$ is the standardized normal distribution function.

In order to obtain the most probable point, an iteration algorithm is proposed by Rackwitz and Fiessler (1978). Recently, the modified algorithm using an adaptive nonlinear approximation with introduction of intermediate variables is proposed in order to achieve rapid convergence by Wang and Grandhi (1994). Also, a nonlinear programming problem is available, Shinozuka (1983).

### Reliability-based Optimization

Reliability-based optimization is an idea of finding optimal structures due to reliability requirements, Frangopol (1985). Recently, a lot of studies on the reliability-based optimization have been conducted, Thoft-Christensen (1990). The problem is commonly formulated as constrained optimization problems as follows:

\begin{align}
\text{minimize} \quad & C \\
\text{subject to} \quad & R \geq R_{\text{min}}
\end{align}

where $C$ is a cost function of the structure and $R_{\text{min}}$ is the minimum reliability required. As a dual problem, the following formulation is also available:

\begin{align}
\text{maximize} \quad & R \\
\text{subject to} \quad & C \leq C_{\text{max}}
\end{align}

where $C_{\text{max}}$ is the maximum allowable structural cost required. This problem finds the design to maximize reliability under the maximum structural cost.

Historically, the reliability-based optimization problem is commonly formulated in the form of Problem (1.1). For example, the reliability-based design of a truss is to find the minimum weight subject to the minimum reliability requirement in terms of member dimensions; i.e., thickness or cross-sectional area, Thoft-Christensen and Murotsu (1986).

In a design problem of a fiber reinforced composite material, however, the structural performance can be changed drastically by designing the stacking sequence, even if the plate thickness is constant. Therefore, two types of problem have been developed in the deterministic design problem. One is to find the maximum structural performance under the
maximum cost requirement. The other is to find the minimum cost under the structural performance requirements. The former formulation is commonly used to investigate the structural performance of the basic structural elements; i.e., beam, plate or shell: The latter is used in the actual applications.

Since the objective of this thesis is to investigate the reliability of the composite plate subject to buckling, the reliability-based design is formulated as Problem (1.2). The reliability of a structure is defined as the generalized system reliability index \( \beta^s = -\Phi^{-1}(P_f) \). Then, the reliability-based design problem can be formulated as:

\[
\begin{align*}
\text{maximize} & \quad \beta^s(x) \\
\text{subject to} & \quad C(x) \leq C_{\text{max}} \\
& \quad x_i^L \leq x_i \leq x_i^U \quad i = 1, \ldots, N
\end{align*}
\]  

(1.3)

where \( x = (x_1, x_2, \ldots, x_n)^T \) is a set of design variables and \( x_i^L \) and \( x_i^U \) are simple lower and upper bounds of \( x_i \), respectively.

**Reliability of Composite Laminated Plate**

Composite material is known to have more uncertainties than a conventional material due to the fabrication process. Effect of the uncertainties on strength of composite structure has been recognized as the most important problem.

Reliability studies for the failure of composite material from the point of macroscopic and microscopic view are described in Cederbaum et al. (1992). From macroscopic standpoint, reliability is evaluated by using Hassin's rule as a failure criteria. While, from microscopic standpoint, failures of the fibers, matrix or fiber/matrix interfaces are considered.

A whole fracture process of unidirectional material, starting from random micro fracture to the final failure, is modeled by Yushanov and Joshi (1995). Matrix crack and single fiber break are considered as primary fracture mechanisms and the interface debonding and the crack penetration into matrix, etc., are considered as secondary fracture mechanisms. The model incorporates the primary and the secondary mechanisms and their interaction.

Reliabilities of composite structures are studied by Park et al. (1996). The structural response is evaluated by stochastic finite element method. Investigated are the bending failure probability of the simply supported plate under uniform pressure and the tensile strength reliability of the laminated composite plate with hole.

Fitzsimmons (1991) studies reliability of composite panel containing a delamination under compression loads. Considering the occurrence probabilities of panel buckling and delamination buckling, the critical delamination size is determined.

The reliability analysis and the reliability-based design under the in-plane strength by the first ply failure criterion have already been studied, Miki et al. (1990), (1992), Shao et al. (1993), and Murotsu et al. (1994). The studies have shown that the reliability
attains higher values as the number of fiber axes is increased and that the reliability-based design approaches a quasi-isotropic laminate construction. The design is very different from the deterministic optimum design of which the orientation angle runs along the loading direction.

In NASA, the computation code IPACS (Integrated Probabilistic Assessment of Composite Structures) is developed, Chamis and Shiao (1992). The code can handle both composite mechanics and composite structures. The composite micromechanics, macromechanics and laminate theory are embedded as the composite mechanics. The structural analysis code can evaluate the buckling loads, stress concentration factors, displacements, and so on. Then, the system reliability is calculated by adaptive importance sampling.

1.3 Objectives

The objective of this thesis is to apply the structural reliability theory and the reliability-based optimization to a composite laminated plate subject to buckling.

As a buckling failure model suitable for the reliability analysis, a series system consisting of eigen modes which correspond to the buckling mode candidates is proposed. Actual buckling mode may be different from the buckling mode at the mean value point design due to variations of design parameters. Therefore, it is necessary to consider several number of eigen modes. However, previous studies about reliability subject to buckling did not consider the possibility of the mode shifting due to uncertainties of design parameters.

Following the formulation of the reliability analysis, effects of the variations in design parameters such as material constants, ply orientation angles and applied loads are investigated on reliability and optimum design.

In reliability-based optimization, the optimum designs which maximize the system reliability are obtained under several conditions. Then, the effects of the number of fiber axes on the reliability is also investigated.

1.4 Contents of Thesis

A preliminary overview of the thesis is given in the following.

In Chapter 2, effects of variations of material constants and ply orientation angle of each ply and the applied loadings on the buckling load are investigated through the buckling sensitivity analysis with respect to these variables. In a practical application, a laminated plate is mainly stacked symmetrically with respect to the mid-plane to avoid bending-extension coupling. However, the actual stacking sequence of the laminated plate will not be symmetric due to the random variations of these variables. Consequently, buckling analysis for an unsymmetric laminate is required. As governing equations of an unsymmetric laminate are very complicated due to the coupling, it is difficult to solve it analytically. Therefore,
an approximation method is utilized. The governing equations are replaced by those of a symmetric laminated plate, where the bending stiffness coefficients are replaced by the reduced bending stiffness. Then, the Galerkin equation for a symmetric laminated plate is directly utilized, which reduces to an eigenvalue problem. The buckling load factor which is the ratio of the buckling load to the applied load is obtained as a minimum eigenvalue. The sensitivity is evaluated by utilizing the derivatives of the eigenvalue. Through numerical calculations, Young’s modulus in the fiber direction and the orientation angle of each ply as well as applied loads are shown to give dominant effects on the buckling load. Moreover, though it is usually known that the outer-most ply has the main effect on the buckling load which is a function of the bending stiffness, it is shown that the fact is not always true due to the effect of the bending-torsion coupling.

Chapter 3 is devoted to the reliability analysis of simply-supported composite laminated plate subject to buckling, where material constants and orientation angle of each ply have some probabilistic variations as well as applied loads. The plate will buckle when the buckling load factor is equal to or less than unity under the applied loads. Therefore, the limit state is defined as a state where the buckling load factor becomes equal to unity. Since the buckling load factor is obtained as a minimum eigenvalue, all the eigenvalues should be larger than unity in order not to buckle the plate. Therefore, the buckling failure is modeled as a series system consisting of eigen modes. The mode reliability is evaluated through the FORM. In the mode reliability searching process, the changes of random variables may cause mode crossings. In order to keep track of the intended mode, a mode tracking strategy which is based on eigenvector orthonormality is used. Consequently, the mode reliability is evaluated correctly, even if the mode order at the design point is not equal to that at the mean value point. The laminate construction of a design point will sometimes be unsymmetric. When the stacking sequence is turned upside down, the reversed laminate construction has the same property as the original laminate. Also, the corresponding point in the $U$-space has the same distance from the origin as a design point. Therefore, the point is also considered as the design point. Finally, the system reliability is evaluated by Ditlevsen’s bounds. Through numerical calculations, the proposed method is shown to have a sufficient accuracy in comparison with Monte Carlo simulation. Then, it is shown that the Young’s modulus in the fiber direction and the orientation angle of each ply as well as applied loads have dominant effect on the reliability.

In Chapter 4, the reliability-based optimum design problem to maximize the buckling reliability in terms of mean orientation angle of each ply is considered. Since the reliability is evaluated by the FORM which can be formulated as a nonlinear programming problem, the reliability-based design problem becomes a nested problem with two levels of optimization. Through numerical calculations, the two types of angle-ply laminates are compared. One is a bi-axial angle-ply with one fiber orientation variable $[+\theta/ -\theta/ -\theta/ +\theta]_s$ and the other is a tetra-axial angle-ply with two orientation variables $[+\theta_1/ -\theta_1/ -\theta_2/ +\theta_2]_s$. The obtained
results show that the reliability is increased as the number of fiber axes increases. This tendency is in good agreement with the reliability-based design subject to in-plane strength. Then, the reliability-based optimum designs are compared with the deterministic buckling load maximization designs under several load conditions. Difference in the ply orientation angle between the surface layer and the mid-plane layer of the reliability-based design is larger than that of the deterministic design. This is because the larger difference reduces variations of bending stiffness due to variations of orientation angles. The mode reliabilities of the reliability-based design are well balanced when the deterministic design has a repeated buckling mode. In this case, the deterministic design has a single dominant failure mode and hence the reliability is much lower. On the other hand, when the deterministic design has a single buckling mode, the difference between the reliability-based design and the deterministic design is small. The buckling mode of the deterministic design corresponds to a single dominant failure mode in the reliability-based design. Therefore, the deterministic design has a relatively higher reliability, because it has the highest value of the limit state function of the single dominant mode.

Chapter 5 is concerned with the effect of correlations between random variables on the reliability and the reliability-based design. In the numerical calculations, three cases of the random variables are investigated. They are correlations between the Young's modulus in the fiber direction of each ply, between the ply orientation angles, and between components of applied loads. In the first and the third cases, the reliability is decreased as the correlations are increased. Moreover, the reliability-based design approaches the deterministic design, as the correlation between applied loads is increased. This is because the variation of the resultant load direction is decreased. On the other hand, the reliability-based design is different from the deterministic design when the correlation between the load is small, that is, the variation of the resultant load direction is not small. Finally, the effect of the correlations between ply orientation angles is different from the above two cases. Since the buckling load is strongly nonlinear in terms of the orientation angle, the reliability has a peak at some positive value of the correlation. Then, the reliability-based design does not reach the deterministic design when the correlation is increased. The above results show that the correlations play an important role in the reliability analysis and the reliability-based designs.

Chapter 6 summarizes the main conclusions of the thesis and describes further topics.

Appendix A describes the sequential programming method used in the reliability analysis and the reliability-based design. In Appendix B, the derivatives of eigenvectors are explained, which is used in the sensitivity analysis for the repeated eigenvalue and the mode tracking. In Appendix C, the probability of correlated normal distribution is illustrated.
Chapter 2

BUCKLING AND SENSITIVITY ANALYSIS

2.1 Introduction

In reliability analysis, a limit state function must be determined to define a failure state of a structure. Buckling load of a laminated plate must be evaluated by considering variations of the material constants or the ply orientation angles. In a practical application, a symmetric laminated plate is mainly used to avoid a bending-extension coupling which makes the structural behavior complicate. However, the actual stacking sequence of a laminated plate will be unsymmetric, even though the mean stacking sequence is symmetric due to the random variations of material properties and orientation angle of each ply. Consequently, it is required to consider an unsymmetric configuration for the buckling analysis.

The FORM can be formulated as a nonlinear programming problem. In order to determine the searching direction of the reliability, derivatives of the limit state function with respect to random variables are required. The sensitivity also gives informations about the effects of variations of the random variables on the buckling load.

In this chapter, classical lamination theory is used as a basic mechanics of a composite laminated plate. Then, the reduced bending stiffness method is introduced as an approximate buckling analysis for an unsymmetric laminated plate, Ashton (1969). Mathematically, the buckling analysis reduces to an eigenvalue problem. Using sensitivity analysis of an eigenvalue problem, sensitivity analyses of the buckling load with respect to the material constants and orientation angle of each ply as well as applied load are formulated. Through numerical calculations, dominant effect on the buckling load is evaluated. Then, effect of the bending-extension coupling on the buckling is also investigated.

2.2 Buckling Analysis

This section starts with a brief introduction of classical lamination theory. Then, the reduced bending stiffness method is introduced as the approximate buckling analysis for
an unsymmetric laminated plate. Replacing the bending stiffness by the reduced bending stiffness, the Galerkin equation for a symmetric laminated plate can be directly utilized. The buckling analysis reduces to an eigenvalue problem. Finally, ordinary buckling analysis for an orthotropic model is applied.

### 2.2.1 Constitutive Equations

Classical lamination theory assumes that orthotropic layers are perfectly bonded together, as shown in Fig. 2.1(a), with a non-shear-deformable and infinitely thin bondline.

The constitutive equations are given as follows, Whitney (1987):

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{22} & A_{26} & B_{22} & B_{26} & B_{26} & B_{26} \\
A_{66} & B_{16} & B_{16} & B_{16} & B_{16} & B_{16} \\
D_{11} & D_{12} & D_{16} & D_{22} & D_{26} & D_{26} & D_{26} \\
\text{symmetry} & \text{symmetry} & \text{symmetry} & \text{symmetry} & \text{symmetry} & \text{symmetry} & \text{symmetry}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\varepsilon_{xy}^0 \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]  

(2.1)
The $A$ and $D$ matrices are extensional and flexural stiffness matrices, respectively. The $A$ matrix relates the in-plane stress resultants $N$ to the mid-plane strains $\varepsilon^0$ while the $D$ matrix does the moment resultants $M$ to the mid-plane curvature $\kappa$. The $B$ matrix, on the other hand, relates the in-plane stress resultants to the curvature and moment resultants to the mid-plane strains, and hence it is called the bending-extension coupling matrix. The $B$ matrix can be avoided by a symmetric placement of the plies with different orientations with respect to the mid-plane of a laminate which is mainly used in the structural applications.

In addition to the bending-extension coupling, certain elements of the $A$, $B$, and $D$ matrices result in coupling response. The $A_{16}$ and $A_{26}$ terms induce shear-extension coupling, and the $D_{16}$ and $D_{26}$ terms induce bending-torsion coupling. Similar to these couplings, bending-shear coupling and extension-torsion coupling results from the $B_{16}$ and $B_{26}$ terms. By proper selection of the stacking sequence, these coupling terms can be eliminated. For example, the shear-extension coupling can be eliminated by using a negative angle ply for every positive angle ply used in the laminate. Such a laminate is called a balanced laminate. The $D_{16}$ and $D_{26}$ terms could be eliminated by stacking the balanced laminate blocks infinitely. In many practical applications, however, these terms are neglected for convenience.

The elements of each matrix are defined as follows:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} \tilde{Q}_{ij}(1, z, z^2) \, dz \quad (i, j = 1, 2, 6) \tag{2.2}$$

where $h$ is the plate thickness and $\tilde{Q}_{ij}$ is the stiffness of each ply. When the material principal axis of the layer $(1-2)$ is rotated to the reference axis $(x-y)$ as shown in Fig. 2.1(b), the stiffnesses $\tilde{Q}_{ij}$ are related to the material invariants $U_i$ and the orientation angle $\theta$ as follows:

$$\begin{pmatrix} \tilde{Q}_{11} \\ \tilde{Q}_{22} \\ \tilde{Q}_{12} \\ \tilde{Q}_{66} \\ \tilde{Q}_{16} \\ \tilde{Q}_{66} \end{pmatrix} = \begin{bmatrix} U_1 & \cos 2\theta & \cos 4\theta \\ U_2 & -\cos 2\theta & \cos 4\theta \\ U_4 & 0 & -\cos 4\theta \\ U_5 & 0 & \cos 4\theta \\ 0 & \sin 2\theta/2 & \sin 4\theta \\ 0 & \sin 2\theta/2 & -\sin 4\theta \end{bmatrix} \begin{pmatrix} 1 \\ U_2 \\ U_3 \end{pmatrix} \tag{2.3}$$

where material invariants $U_i (i = 1, \ldots, 5)$ are given in terms of the unidirectional material:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & 3 & 2 & 4 \\ 4 & -4 & 0 & 0 \\ 1 & 1 & -2 & -4 \\ 1 & 1 & 3 & -4 \\ 1 & 1 & -1 & 4 \end{bmatrix} \begin{pmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{pmatrix} \tag{2.4}$$

The stiffness of the unidirectional material is given in terms of four independent engineering
Figure 2.2: Laminated plate composed of an even number of plies.

material constants:
\[ Q_{11} = \frac{E_1}{1 - \nu_1^2 E_2 / E_1}, \quad Q_{12} = \frac{\nu_1 E_2}{1 - \nu_1^2 E_2 / E_1}, \]
\[ Q_{22} = \frac{E_2}{1 - \nu_1^2 E_2 / E_1}, \quad Q_{66} = E_6 \quad (2.5) \]

where \( E_1, E_2, \nu_1 \) and \( E_6 \) denote Young’s moduli in the fiber direction 1 and the lateral direction 2, Poisson’s ratio and the shear modulus, respectively.

2.2.2 Stiffness of Laminated Plate

The stiffness matrices \( A, B, \) and \( D \) of a laminated plate are given in Eq. (2.2). Substituting Eqs. (2.3) and (2.4) into Eq. (2.2), each term is written in the following form:

\[ [A_{11}, B_{11}, D_{11}] = \int_{-h/2}^{h/2} U_1 [1, z, z^2] + U_2 [1, z, z^2] \cos 2\theta + U_3 [1, z, z^2] \cos 4\theta \, dz \]
\[ [A_{22}, B_{22}, D_{22}] = \int_{-h/2}^{h/2} U_1 [1, z, z^2] - U_2 [1, z, z^2] \cos 2\theta + U_3 [1, z, z^2] \cos 4\theta \, dz \]
\[ [A_{12}, B_{12}, D_{12}] = \int_{-h/2}^{h/2} U_4 [1, z, z^2] - U_3 [1, z, z^2] \cos 4\theta \, dz \quad (2.6) \]
\[ [A_{66}, B_{66}, D_{66}] = \int_{-h/2}^{h/2} U_6 [1, z, z^2] - U_3 [1, z, z^2] \cos 4\theta \, dz \]
\[ [A_{16}, B_{16}, D_{16}] = \int_{-h/2}^{h/2} U_2 [1, z, z^2] \sin 2\theta + U_3 [1, z, z^2] \sin 4\theta \, dz \]
\[ [A_{26}, B_{26}, D_{26}] = \int_{-h/2}^{h/2} U_2 [1, z, z^2] \sin 2\theta - U_3 [1, z, z^2] \sin 4\theta \, dz \]

Consider the laminated plate which consists of \( n \) (where \( n = \text{even} \)) plies whose thicknesses are \( h_0 \) each. The total ply thickness is \( h = nh_0 \). Introducing a new variable \( t \) which
indicates a lamination order as shown in Fig. 2.2, the stiffness terms can be written in the summation form:

\[ [A_{11}, B_{11}, D_{11}] = \left[ h_0, \frac{h_0^2}{2}, \frac{h_0^5}{3} \right] \sum_{t=1-n/2}^{n/2} \left( t^{(t)} + U_2^{(t)} \cos 2\theta_t + U_3^{(t)} \cos 4\theta_t \right) \left[ 1, t^2 - (t - 1)^2, t^3 - (t - 1)^3 \right] \]

\[ [A_{22}, B_{22}, D_{22}] = \left[ h_0, \frac{h_0^2}{2}, \frac{h_0^5}{3} \right] \sum_{t=1-n/2}^{n/2} \left( t^{(t)} - U_2^{(t)} \cos 2\theta_t + U_3^{(t)} \cos 4\theta_t \right) \left[ 1, t^2 - (t - 1)^2, t^3 - (t - 1)^3 \right] \]

\[ [A_{12}, B_{12}, D_{12}] = \left[ h_0, \frac{h_0^2}{2}, \frac{h_0^5}{3} \right] \sum_{t=1-n/2}^{n/2} \left( U_4^{(t)} - U_3^{(t)} \cos 4\theta_t \right) \left[ 1, t^2 - (t - 1)^2, t^3 - (t - 1)^3 \right] \]

\[ [A_{66}, B_{66}, D_{66}] = \left[ h_0, \frac{h_0^2}{2}, \frac{h_0^5}{3} \right] \sum_{t=1-n/2}^{n/2} \left( U_5^{(t)} - U_3^{(t)} \cos 4\theta_t \right) \left[ 1, t^2 - (t - 1)^2, t^3 - (t - 1)^3 \right] \]

\[ [A_{16}, B_{16}, D_{16}] = \left[ h_0, \frac{h_0^2}{2}, \frac{h_0^5}{3} \right] \sum_{t=1-n/2}^{n/2} \left( \frac{U_2^{(t)}}{2} \sin 2\theta_t + U_3^{(t)} \sin 4\theta_t \right) \left[ 1, t^2 - (t - 1)^2, t^3 - (t - 1)^3 \right] \]

\[ [A_{26}, B_{26}, D_{26}] = \left[ h_0, \frac{h_0^2}{2}, \frac{h_0^5}{3} \right] \sum_{t=1-n/2}^{n/2} \left( \frac{U_2^{(t)}}{2} \sin 2\theta_t - U_3^{(t)} \sin 4\theta_t \right) \left[ 1, t^2 - (t - 1)^2, t^3 - (t - 1)^3 \right] \]

where the superscript \((t)\) of \(U_t\) corresponds to the \(t\)-th ply.

### 2.2.3 Reduced Bending Stiffness Method

Since the unsymmetric laminate has bending-extension coupling, it is difficult to directly solve the governing equation. Therefore, the reduced bending stiffness method, Ashton (1969), is used here as an approximation method. The method reduces the problem to an equivalent anisotropic bending problem without bending-extension coupling. Then, the Galerkin method for the analysis of symmetric laminate plate can be directly applied, Whitney (1987).

The constitutive equation of the unsymmetric laminate, Eq. (2.1), is expressed as follows:

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} = 
\begin{bmatrix}
A & B \\
B & D
\end{bmatrix} 
\begin{bmatrix}
\varepsilon^0 \\
\kappa
\end{bmatrix}
\]

Multiplying the first equation of Eq. (2.8) by \(A^{-1}\) yields:

\[
\varepsilon^0 = A^{-1}N - A^{-1}B\kappa
\]

Then, substituting the result into the second results in

\[
M = BA^{-1}N + (D - BA^{-1}B)\kappa
\]
Now, Eq. (2.8) can be written in the following equation:

$$\begin{bmatrix} \varepsilon^0 \\ M \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ (-B^*)^T & D^* \end{bmatrix} \begin{bmatrix} N \\ \kappa \end{bmatrix}$$

(2.9)

where

$$A^* = A^{-1}, \quad B^* = -A^{-1}B, \quad D^* = D - BA^{-1}B$$

The strain energy $U$ of a laminated rectangular plate can be written in the form:

$$U = \frac{1}{2} \int_0^a \int_0^b \left( N_x \varepsilon_x^0 + N_y \varepsilon_y^0 + N_{xy} \varepsilon_{xy}^0 + M_x \kappa_x + M_y \kappa_y + M_{xy} \kappa_{xy} \right) \, dx \, dy$$

(2.10)

where $a$ and $b$ are the plate dimensions of $x$ and $y$ directions, respectively. Using the constitutive relation of the plate in the form of Eq. (2.9), Eq. (2.10) becomes:

$$U = \frac{1}{2} \int_0^a \int_0^b \left( A_{11}^* N_x^2 + A_{22}^* N_y^2 + A_{66}^* N_{xy}^2 + 2 A_{12}^* N_x N_y + 2 A_{16}^* N_x N_{xy} + 2 A_{26}^* N_y N_{xy} \\
+ D_{11}^* \kappa_x^2 + D_{22}^* \kappa_y^2 + D_{66}^* \kappa_{xy}^2 + 2 D_{12}^* \kappa_x \kappa_y + 2 D_{16}^* \kappa_x \kappa_{xy} + 2 D_{26}^* \kappa_y \kappa_{xy} \right) \, dx \, dy$$

(2.11)

There are no bending-extension coupling terms appearing in Eq. (2.11). This suggests that an unsymmetric laminated plate problem can be solved in an uncoupled form:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix}$$

(2.12)

The unsymmetric laminate can be modeled as a symmetric laminate with bending stiffness of $D^*$ instead of $D$. Thus, the governing equation of a symmetric laminated plate can be utilized by substituting $D_{ij}^*$ for the bending stiffness $D_{ij}$.

### 2.2.4 Galerkin Method

Consider a rectangular laminated plate with dimensions $a$ and $b$ subject to uniform biaxial compression and shear loads $(N_x, N_y, N_{xy})$, as shown in Fig. 2.1(a). The governing equation of the plate is represented as follows:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4}$$

$$= N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}$$

(2.13)

When the out-of-plane displacement $w$ is sought in the form:

$$w = \sum_{m=1}^{M} \sum_{n=1}^{N} \phi_{mn} W_{mn}(x, y)$$

where $\phi_{mn}$ is undetermined coefficients and the function $W_{mn}$ is a modal function of $x$ and $y$ which will satisfy the geometrical boundary conditions. Then, the Galerkin equation is
written in the following form:

\[
\int \int \left[ \left\{ D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} \right\} 
- \left( N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \right] W_{mn}(x,y) \, dx \, dy
\]

(2.14)

\[
- \int_{S_x} \left[ M_x \frac{\partial W_{mn}}{\partial x}(S_x, y) - \left( \frac{\partial M_x}{\partial x} + 2 \frac{\partial M_{xy}}{\partial y} + N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) W_{mn}(S_x, y) \right] \, dy
\]

\[
+ \int_{S_y} \left[ M_y \frac{\partial W_{mn}}{\partial y}(x, S_y) - \left( 2 \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} + N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} \right) W_{mn}(x, S_y) \right] \, dx = 0
\]

where \( S_x \) is defined along the edges \( x = \text{constant} \) and \( S_y \) along the edges \( y = \text{constant} \).

For a simply supported plate, the following boundary conditions are applicable.

At \( x = 0, a \): \( w = 0, M_x = D_{11} \frac{\partial^2 w}{\partial x^2} + 2D_{16} \frac{\partial w}{\partial x \partial y} + D_{22} \frac{\partial w}{\partial y^2} = 0 \) (2.15)

At \( y = 0, b \): \( w = 0, M_y = D_{12} \frac{\partial^2 w}{\partial x^2} + 2D_{26} \frac{\partial w}{\partial x \partial y} + D_{22} \frac{\partial w}{\partial y^2} = 0 \) (2.16)

The displacement \( w \) satisfying the geometrical boundary condition is assumed to be expressed in the double sine series:

\[
w = \sum_{m=1}^{M} \sum_{n=1}^{N} \phi_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

(2.17)

where \( W_{mn}(x,y) = \sin(m\pi x/a) \sin(n\pi y/b) \). For the assumed series (2.17), the Galerkin equation (2.14) yields the following equations:

\[
\int_0^a \int_0^b \left[ \left\{ D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} \right\}
- \left( N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy
\]

\[
+ 2D_{26} \int_0^a \left[ (-1)^n \left( \frac{\partial^2 w}{\partial x \partial y} \right)_{y=b} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)_{y=0} \right] \frac{n\pi}{b} \sin \frac{m\pi x}{a} \, dx
\]

\[
+ 2D_{16} \int_0^b \left[ (-1)^m \left( \frac{\partial^2 w}{\partial x \partial y} \right)_{x=a} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)_{x=0} \right] \frac{m\pi}{a} \sin \frac{n\pi y}{b} \, dy
\]

(2.18)

\[
= 0 \quad \{ m = 1, \ldots, M \}
\]

\[
\{ n = 1, \ldots, N \}
\]

Substituting Eq. (2.17) into Eq. (2.18), the following set of algebraic equations is obtained:

\[
\pi^4 \left[ D_{11} m^4 + 2(D_{12} + D_{66}) m^2 n^2 R^2 + D_{22} n^4 R^4 + N_x \frac{m^2 a^2}{\pi^2} + N_y \frac{n^2 R^2 a^2}{\pi^2} \right] \phi_{mn}
- 32\pi^2 mnR \sum_{i=1}^{M} \sum_{j=1}^{N} M_{ij} \left[ (m^2 + i^2) D_{16} + (n^2 + j^2) D_{26} R^2 + N_{xy} \frac{a^2}{\pi^2} \right] \phi_{ij} = 0
\]

(2.19)
where \( R \) is an aspect ratio \( a/b \) and

\[
M_{ij} = \begin{cases} 
\frac{ij}{(m^2 - i^2)(n^2 - j^2)} & \text{if } m + i = \text{odd and } n + j = \text{odd} \\
0 & \text{otherwise}
\end{cases} \tag{2.20}
\]

Eq. (2.19) yields a set of \( M \times N \) homogeneous equations. This set of equations can be reduced to an eigenvalue equation for \( \phi_{mn} \), when the stiffness terms and the loading terms are separated.

The eigenvalue equation is written as follows:

\[
[K - \lambda K_G]\phi = 0 \tag{2.21}
\]

where \( \phi \) is a vector consisting of \( \phi_{mn} \), \( K \) the stiffness matrix and \( K_G \) the geometry stiffness matrix. The elements of each matrix are written as follows:

\[
K_{mn} = \pi^4 \left[ D_{11} m^4 + 2(D_{12} + 2D_{06})m^2 n^2 R^2 + D_{22} n^4 R^4 \right] \tag{2.22}
\]

\[
K_{mn}^{ij} = -32\pi^2 m n R M_{ij} \left[ (m^2 + t^2)D_{16} + (n^2 + j^2)D_{26} R^2 \right] \tag{2.23}
\]

\[
K_{mn} = \pi^2 a^2 \left( m^2 N_x + n^2 R^2 N_y \right) \tag{2.24}
\]

\[
K_{mn}^{ij} = 32a^2 m n R M_{ij} N_{xy} \tag{2.25}
\]

The subscripts \( m, n, i \) and \( j \) correspond to that of Eq. (2.19). Eqs. (2.22) and (2.24) are diagonal elements of each matrix, while Eqs. (2.23) and (2.25) are off-diagonal elements.

The minimum eigenvalue \( \lambda_{\min} \) corresponds to the buckling load factor subject to the applied load \( (N_x, N_y, N_{xy}) \). That is, the plate will buckle when the load reaches \( \lambda_{\min} \times (N_x, N_y, N_{xy}) \). The resulting equations can be separated into two sets, one for \( m + n = \text{even} \) and the other for \( m + n = \text{odd} \). Therefore, the buckling load factor corresponds to the smaller one of the two minimum eigenvalues; one for \( m + n = \text{even} \) and the other for \( m + n = \text{odd} \).

### 2.2.5 Notes on Buckling Analysis

The stiffness matrix \( K \) is always positive definite. On the other hand, the geometry stiffness matrix \( K_G \) is not always positive definite, depending on an applied loading condition.

For most computational algorithms of a general symmetric eigenvalue problem with the form of \( [A - \lambda B]\phi = 0 \), the matrix \( B \) must be positive definite, Watanabe et al. (1989). Therefore, the eigenvalue equation is transposed to the following form:

\[
[K_G - \mu K]\phi = 0
\]

where \( \mu = 1/\lambda \). The buckling load is obtained as reciprocal of the maximum eigenvalue \( \lambda_{\min} = 1/\mu_{\max} \).
2.2.6 Orthotropic Modeling

When the plate is modeled as an orthotropic plate which ignores the bending-torsion coupling terms, $D_{16}$ and $D_{26}$, the buckling load subject to a biaxial compression load is obtained from the following equation, e.g., Whitney (1987):

$$
\lambda = \min_{m,n} \frac{\pi^2 [D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2R^2 + D_{22}n^4R^4]}{a^2(m^2N_z + n^2R^2N_y)} \tag{2.26}
$$

This equation corresponds to the buckling eigenvalue equation (2.21) without off-diagonal elements (see Eqs. (2.22) and (2.24)).

This equation is widely used for an approximate analysis in practical applications or deterministic optimization problems, Haftka and Gürdal (1992).

However, the objective of the thesis is to evaluate the effect of the variations of design parameters on the buckling. For this purpose, the buckling load of the perturbed design must be evaluated as accurate as possible. Therefore, the Galerkin method together with the reduced bending stiffness method considering the bending-extension coupling matrix is used for the buckling analysis.

2.3 Sensitivity Analysis of Eigenvalue Problem

As the buckling analysis reduces to an eigenvalue problem, sensitivity analysis of the buckling load is obtained through the sensitivity analysis of an eigenvalue problem. In this section, the sensitivity analysis of an eigenvalue problem is introduced.

The eigenvalue equation of size $n$ is written as follows:

$$
K\phi_j = \lambda_j M\phi_j, \quad j = 1, \ldots, n \tag{2.27}
$$

where $K$ and $M$ are real symmetric matrices, and $\lambda_j$ and $\phi_j$ are the $j$-th eigenvalue and eigenvector, respectively. When the matrix $M$ is positive definite, Eq. (2.27) has $n$ real positive solutions (eigenvalue $\lambda_j$ and corresponding eigenvector $\phi_j$). Then, eigenvalues are assumed to be $M$-orthogonal:

$$
\phi_j^T M \phi_k = \delta_{jk}, \quad j, k = 1, \ldots, n \tag{2.28}
$$

where $\delta_{jk}$ denotes Kronecker's delta.

The sensitivity analysis depends on the state of eigenvalues; single or repeated. When eigenvalues are repeated, any linear combination of the corresponding eigenvalues also satisfies the eigenvalue equation. Therefore, additional information is required to evaluate the sensitivity.

At first, sensitivity analysis of a single eigenvalue is described. Then, sensitivity analysis of the repeated eigenvalues is introduced.
2.3.1 Sensitivity of Single Eigenvalues

Derivatives of eigenvalue with respect to design variables $x_i$, ($i = 1, \ldots, I$) are introduced. Here, matrices $K$ and $M$ are assumed to be a smooth function of design variable $x_i$.

When $\lambda_j$ is a single eigenvalue, the derivative of Eq. (2.27) with respect to design variable $x_i$ is written as follows:

$$\frac{\partial K}{\partial x_i} \phi_j + (K - \lambda_j M) \frac{\partial \phi_j}{\partial x_i} = \frac{\partial \lambda_j}{\partial x_i} M \phi_j + \lambda_j \frac{\partial M}{\partial x_i} \phi_j, \quad i = 1, \ldots, I \tag{2.29}$$

Premultiplying $\phi_j^T$, Eq. (2.29) becomes:

$$\phi_j^T \frac{\partial K}{\partial x_i} \phi_j + \phi_j^T (K - \lambda_j M) \frac{\partial \phi_j}{\partial x_i} = \phi_j^T \left( \frac{\partial \lambda_j}{\partial x_i} M \phi_j + \lambda_j \frac{\partial M}{\partial x_i} \phi_j \right), \quad i = 1, \ldots, I$$

From Eq. (2.27), the second term of the left hand side is equal to 0, and the first term of the right hand side becomes $\frac{\partial \lambda_j}{\partial x_i}$ from Eq. (2.28). Then, the derivative of the single eigenvalue $\lambda_j$ is written as follows, Nakagiri and Hisada (1985):

$$\frac{\partial \lambda_j}{\partial x_i} = \phi_j^T \left( \frac{\partial K}{\partial x_i} - \lambda_j \frac{\partial M}{\partial x_i} \right) \phi_j, \quad i = 1, \ldots, I \tag{2.30}$$

2.3.2 Notes on Numerical Calculation

For the numerical calculation, the eigenvalue equation is transposed to the following form as shown in Section 2.2.5:

$$[K_G - \mu K] \phi = 0$$

where $\mu = 1/\lambda$. The buckling load is obtained as reciprocal of the maximum eigenvalue $\lambda_{\text{min}} = 1/\mu_{\text{max}}$. Then, the derivative of $\mu_j$ is given by:

$$\frac{\partial \mu_j}{\partial x_i} = \phi_j^T \left( \frac{\partial K_G}{\partial x_i} - \mu_j \frac{\partial K}{\partial x_i} \right) \phi_j \tag{2.31}$$

Therefore, the derivative of $\lambda_j$ is:

$$\frac{\partial \lambda_j}{\partial x_i} = -\frac{\partial \mu_j}{\partial x_i} \frac{1}{\mu_j^2} = -\lambda_j^2 \phi_j^T \left( \frac{\partial K_G}{\partial x_i} - \lambda_j \frac{\partial K}{\partial x_i} \right) \phi_j \tag{2.32}$$

2.3.3 Sensitivity of Repeated Eigenvalues

When the solution to Eq. (2.27) produces $N$ repeated eigenvalues $\lambda_i = \lambda_j, i, j = 1, \ldots, N$, computation of derivatives of the eigenvalues is not straightforward. The complication is related to a fact that the eigenvectors $\Phi$ of the repeated eigenvalues are not unique. That is, an infinite number of linear combinations of the eigenvectors also satisfy Eq. (2.27).
A new eigenvector $\tilde{\Phi}_i$ is introduced, which is a linear combination of the originally obtained eigenvectors $\Phi_i$:

$$\tilde{\Phi}_i = \Phi a_i = [\phi_1|\phi_2|\cdots|\phi_N] \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{iN} \end{pmatrix}, \quad (i = 1, \cdots, N)$$

(2.34)

where $a_i$ is an unknown coefficient vector. From the $M$-orthonormality, the vector $a_i$ should satisfy the following equation:

$$a_i^T \Phi^T M \Phi a_j = a_i^T a_j = \delta_{ij}, \quad (i = 1, \cdots, N)$$

(2.35)

Therefore, the vector $a_i$ must be an orthonormal vector.

For the repeated eigenvalues, the derivative of Eq. (2.27) with respect to a design variable $x_j$ is written as follows:

$$\left( \frac{\partial K}{\partial x_j} - \lambda_i \frac{\partial M}{\partial x_j} - \frac{\partial \lambda_i}{\partial x_j} M \right) \Phi a_i + (K - \lambda_i M) \frac{\partial \tilde{\Phi}_i}{\partial x_j} = 0, \quad (i = 1, \cdots, N)$$

(2.36)

Premultiplying by $\Phi^T$ gives:

$$\left[ \Phi^T \left( \frac{\partial K}{\partial x_j} - \lambda_i \frac{\partial M}{\partial x_j} \right) \Phi \Phi^T \right] a_i + \Phi^T (K - \lambda_i M) \frac{\partial \tilde{\Phi}_i}{\partial x_j} = 0, \quad (i = 1, \cdots, N)$$

where the second term is set to zero, since $\Phi^T (K - \lambda_i M)$ is the transpose of the eigenvalue problem solution. From the eigenvector orthonormality, the terms $\Phi^T M \Phi$ is equal to the unit matrix $I$. Therefore,

$$\left[ \Phi^T \left( \frac{\partial K}{\partial x_j} - \lambda_i \frac{\partial M}{\partial x_j} \right) \Phi \right] a_i = 0, \quad (i = 1, \cdots, N)$$

(2.37)

Thus, a subeigenvalue problem in $\partial \lambda_i / \partial x_j$ and $a_i$ is formulated. Unless the solution of the subeigenvalue problem $\partial \lambda_i / \partial x_j$ is also repeated, a unique coefficient matrix $a_i$ is obtained by solving this equation, Ojavlo (1988).

In case where $\partial \lambda_i / \partial x_j$ is repeated, more complicated conditions are required. However, such a case seldom happens and hence it is ignored in this thesis.

### 2.4 Sensitivity Analysis of Buckling Load

In the reliability analysis, material constants $E_1$, $E_2$, $E_6$ and $\nu_1$ of each ply, the ply orientation angle $\theta_i$ and applied load $N_x$, $N_y$ and $N_{xy}$ will be treated as random variables. In this section, the sensitivity analysis of the buckling load factor with respect to these variables are formulated by utilizing the sensitivity analysis of eigenvalues described above.

To obtain the sensitivity of the buckling load, the derivatives of the matrices $K$ and $K_G$ are required as shown in Eq. (2.32).
2.4.1 Sensitivity with Respect to Applied Load

The sensitivity of the geometry stiffness matrix $K_G$ with respect to the applied load is introduced. Each element of $K_G$ is a linear function of applied loads as shown in Eqs. (2.24) and (2.25). Therefore, the sensitivity of each element is constant and given by:

$$\frac{\partial K_{Gmn}}{\partial N_x} = \pi^2 a^2 m^2, \quad \frac{\partial K_{Gmn}}{\partial N_y} = \pi^2 a^2 n^2 R^2, \quad \frac{\partial K_{Gij}}{\partial N_{xy}} = 32a^2 mnRM_{ij} \quad (2.38)$$

As the sensitivity of stiffness matrix with respect to the applied load is equal to zero, the eigenvalue sensitivity (2.32) is written as follows:

$$\frac{\partial \lambda_j}{\partial N_i} = \mu_j^2 \left( \frac{\partial K_G}{\partial N_i} \right) \phi_j = \lambda_j^2 \phi_j^T \left( \frac{\partial K_G}{\partial N_i} \right) \phi_j \quad (2.39)$$

where $N_i$ corresponds to each load element $N_x, N_y$ and $N_{xy}$.

2.4.2 Derivatives of Stiffness Matrix

Sensitivity of the stiffness matrix $K$ is introduced here. As already shown in Eqs. (2.22) and (2.23), each element of the stiffness matrix is a linear combination of the bending stiffness $D_{ij}$. Moreover, the bending stiffness is a function of the material constants and the ply orientation angles.

The derivative of each element of the stiffness matrix $K$ with respect to basic variable $x_k$ is written as follows:

$$\frac{\partial K_{mn}}{\partial x_k} = \pi^4 \left[ m^4 \frac{\partial D_{11}}{\partial x_k} + 2 \left( \frac{\partial D_{12}}{\partial x_k} + \frac{\partial D_{66}}{\partial x_k} \right) m^2 n^2 R^2 + \frac{\partial D_{22}}{\partial x_k} n^4 R^4 \right] \quad (2.40)$$

$$\frac{\partial K^i_{mn}}{\partial x_k} = 32mnR\pi^2 M_{ij} \left[ (m^2 + t^2) \frac{\partial D_{16}}{\partial x_k} + (n^2 + j^2) \frac{\partial D_{66}}{\partial x_k} n^4 R^2 \right] \quad (2.41)$$

2.4.3 Derivatives of Bending Stiffness

The derivative of the flexural stiffness with respect to material constants and the ply orientation angle is introduced. As shown in Eq. (2.7), the flexural stiffness is a function of material invariants $U_i^{(i)}$ and the orientation angle $\theta_i$.

The derivatives with respect to the orientation angle of the $i$-th ply are written as follows:

$$\frac{\partial D_{11}}{\partial \theta_i} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( -2U_2^{(i)} \sin 2\theta_i - 4U_3^{(i)} \sin 4\theta_i \right)$$

$$\frac{\partial D_{22}}{\partial \theta_i} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( 2U_2^{(i)} \sin 2\theta_i - 4U_3^{(i)} \sin 4\theta_i \right)$$

$$\frac{\partial D_{12}}{\partial \theta_i} = \frac{4h_3}{3} \left\{ t^3 - (t-1)^3 \right\} U_3^{(i)} \sin 4\theta_i$$

$$\frac{\partial D_{66}}{\partial \theta_i} = \frac{4h_3}{3} \left\{ t^3 - (t-1)^3 \right\} U_3^{(i)} \sin 4\theta_i \quad (2.42)$$
2.4. SENSITIVITY ANALYSIS OF BUCKLING LOAD

\[
\frac{\partial D_{16}}{\partial \theta_t} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( U_2^{(t)} \cos 2\theta_t + 4U_3^{(t)} \cos 4\theta_t \right)
\]

\[
\frac{\partial D_{26}}{\partial \theta_t} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( U_2^{(t)} \cos 2\theta_t - 4U_3^{(t)} \cos 4\theta_t \right)
\]

Further, the derivatives of material constants of the \( t \)-th ply are given as follows:

\[
\frac{\partial D_{11}}{\partial x_t} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( \frac{\partial U_1^{(t)}}{\partial x_t} + \frac{\partial U_2^{(t)}}{\partial x_t} \cos 2\theta_t + \frac{\partial U_3^{(t)}}{\partial x_t} \cos 4\theta_t \right)
\]

\[
\frac{\partial D_{22}}{\partial x_t} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( \frac{\partial U_1^{(t)}}{\partial x_t} - \frac{\partial U_2^{(t)}}{\partial x_t} \cos 2\theta_t + \frac{\partial U_3^{(t)}}{\partial x_t} \cos 4\theta_t \right)
\]

\[
\frac{\partial D_{12}}{\partial x_t} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( \frac{\partial U_4^{(t)}}{\partial x_t} - \frac{\partial U_3^{(t)}}{\partial x_t} \cos 4\theta_t \right)
\]

\[
\frac{\partial D_{26}}{\partial x_t} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( \frac{1}{2} \frac{\partial U_2^{(t)}}{\partial x_t} \sin 2\theta_t + \frac{\partial U_3^{(t)}}{\partial x_t} \sin 4\theta_t \right)
\]

\[
\frac{\partial D_{26}}{\partial x_t} = \frac{h_3}{3} \left\{ t^3 - (t-1)^3 \right\} \left( \frac{1}{2} \frac{\partial U_2^{(t)}}{\partial x_t} \sin 2\theta_t - \frac{\partial U_3^{(t)}}{\partial x_t} \sin 4\theta_t \right)
\]

where \( x_t \) is representative of \( E_1, E_2, E_6 \) and \( \nu_1 \).

2.4.4 Derivatives of Material Invariants

The derivatives of material invariants \( U_i \) with respect to material constants \( E_1, E_2, E_6 \) and \( \nu_1 \) are introduced. From Eqs. (2.4) and (2.5), \( U_i \) can be described by using material constants:

\[
U_1 = \frac{E_1(3E_1 + 3E_2 + 2\nu_1E_2)}{8(E_1 - \nu_1^2E_2)} + \frac{E_6}{2}
\]

\[
U_2 = \frac{E_1(E_1 - E_2)}{2(E_1 - \nu_1^2E_2)}
\]

\[
U_3 = \frac{E_1(E_1 + E_2 - 2\nu_1E_2)}{8(E_1 - \nu_1^2E_2)} - \frac{E_6}{2}
\]

\[
U_4 = \frac{E_1(E_1 + E_2 + 6\nu_1E_2)}{8(E_1 - \nu_1^2E_2)} - \frac{E_6}{2}
\]

\[
U_6 = \frac{E_1(E_1 + E_2 - 2\nu_1E_2)}{8(E_1 - \nu_1^2E_2)} + \frac{E_6}{2}
\]

Therefore, the derivatives of \( U_i \) are written as follows:

\[
\frac{\partial U_1}{\partial E_1} = \frac{3E_1^2 - 6\nu_1^2E_1E_2 - 2\nu_1^3E_2^2 - 3\nu_1^2E_2^2}{8(E_1 - \nu_1^2E_2)^2}
\]

\[
\frac{\partial U_2}{\partial E_1} = \frac{E_1^2 - 2\nu_1^2E_1E_2 + \nu_1^2E_2^2}{2(E_1 - \nu_1^2E_2)^2}
\]
\[
\frac{\partial U_3}{\partial E_1} = \frac{(E_1 - \nu_1 E_2)(E_1 + \nu_1 E_2 - 2\nu_1^2 E_2)}{8(E_1 - \nu_1^2 E_2)^2} \\
\frac{\partial U_4}{\partial E_1} = \frac{E_1^2 - 2\nu_1^2 E_1 E_2 - 6\nu_1^3 E_2^2 - \nu_1^2 E_2^2}{8(E_1 - \nu_1^2 E_2)^2} \\
\frac{\partial U_5}{\partial E_1} = \frac{(E_1 - \nu_1 E_2)(E_1 + \nu_1 E_2 - 2\nu_1^2 E_2)}{8(E_1 - \nu_1^2 E_2)^2} = \frac{\partial U_3}{\partial E_1} \\
\]

\[
\frac{\partial U_1}{\partial E_2} = \frac{E_1^2(3 + 2\nu_1 + 3\nu_1^2)}{8(E_1 - \nu_1^2 E_2)^2} \\
\frac{\partial U_2}{\partial E_2} = \frac{E_1^2(\nu_1^2 - 1)}{2(E_1 - \nu_1^2 E_2)^2} \\
\frac{\partial U_3}{\partial E_2} = \frac{E_1^2(1 - \nu_1^2)}{8(E_1 - \nu_1^2 E_2)^2} \\
\frac{\partial U_4}{\partial E_2} = \frac{E_1^2(1 + 6\nu_1 + \nu_1^2)}{8(E_1 - \nu_1^2 E_2)^2} \\
\frac{\partial U_5}{\partial E_2} = \frac{E_1^2(1 - \nu_1^2)}{8(E_1 - \nu_1^2 E_2)^2} = \frac{\partial U_3}{\partial E_2} \\
\frac{\partial U_1}{\partial E_2} = \frac{1}{2}, \quad \frac{\partial U_2}{\partial E_2} = 0, \quad \frac{\partial U_3}{\partial E_2} = \frac{1}{2}, \quad \frac{\partial U_4}{\partial E_2} = \frac{1}{2}, \quad \frac{\partial U_5}{\partial E_2} = 0 \quad (2.47)
\]

Substituting these derivatives into Eqs. (2.40) and (2.41), the derivative of the stiffness matrix \( K \) is obtained. Then, the sensitivity of the buckling load factor is calculated by Eq. (2.32).

### 2.4.5 Derivatives of Reduced Bending Stiffness

For an unsymmetric laminated plate, the reduced bending stiffness method is used. Then, the flexural stiffness \( D \) is permuted by the reduced bending stiffness \( D^* \). Therefore, Eqs. (2.40) and (2.41) become:

\[
\frac{\partial K_{mn}}{\partial x_k} = \pi^4 \left[ m^4 \frac{\partial D_{11}}{\partial x_k} + 2 \left( \frac{\partial D_{12}}{\partial x_k} + 2 \frac{\partial D_{26}}{\partial x_k} \right) m^2 n^2 R^2 + \frac{\partial D_{12}^*}{\partial x_k} n^4 R^4 \right] \\
\frac{\partial K_{mn}^i}{\partial x_k} = 32 \pi^2 M_{ij} \left[ (m^2 + n^2) \frac{\partial D_{18}^*}{\partial x_k} + (n^2 + j^2) \frac{\partial D_{28}^*}{\partial x_k} R^2 \right] \\
\]

(2.49) (2.50)
The derivatives of $D^*$ are calculated by the following chain rule:

$$\frac{\partial D^*}{\partial x_i} = \frac{\partial D}{\partial x_i} - 2 \frac{\partial B}{\partial x_i} A^{-1} B - B \frac{\partial A^{-1}}{\partial x_i} B$$  \hspace{1cm} (2.51)

The derivative of the coupling matrix $B$ is obtained by the similar method as Eqs. (2.42) and (2.43). From Eq. (2.2), the coefficients of Eqs. (2.42) and (2.43) are changed as follows:

$$\frac{h_0^3}{3} \{ t^3 - (t - 1)^3 \} \rightarrow \frac{h_0^2}{2} \{ t^2 - (t - 1)^2 \}$$  \hspace{1cm} (2.52)

The derivative of $A^{-1}$ is obtained by using $AA^{-1} = I$.

$$\frac{\partial A^{-1}}{\partial x_i} = -A^{-1} \frac{\partial A}{\partial x_i} A^{-1}$$  \hspace{1cm} (2.53)

The derivative of the extensional stiffness $A$ can be evaluated by using the similar way as that of $D$ or $B$. The coefficients of Eqs. (2.42) and (2.43) are changed as follows:

$$\frac{h_0^3}{3} \{ t^3 - (t - 1)^3 \} \rightarrow h_0$$  \hspace{1cm} (2.54)

## 2.5 Numerical Calculations

Numerical examples are given for an 8 ply angle-ply laminated plate $[(+\theta/-\theta)_2]_S$ made up with graphite/epoxy (T300/5208); $E_1 = 181.0$ (GPa), $E_2 = 10.3$ (GPa), $E_3 = 7.17$ (GPa), and $\nu_1 = 0.28$. In buckling analysis, the number of series in the Galerkin equation is set to $M = N = 10$. Then, the matrix size of two eigen equations for $m + n = \text{even}$ and $m + n = \text{odd}$ is $50 \times 50$.

In order to investigate the effect of the laminate construction, the buckling load $\lambda$ is standardized in terms of plate dimensions (length $a$, width $b$ and thickness $h$) as follows, Grenestedt (1990) and Miki and Sugiyama (1993):

$$\lambda^* = \frac{12a^2\lambda}{\pi^2 R^3 h^3}$$  \hspace{1cm} (2.55)

### 2.5.1 Effect of Eigen Mode

Consider a square plate (aspect ratio $R = 1$) subject to uniaxial compression load ($N_x = 1$ (GPa), $N_y = N_{xy} = 0$). Buckling load and lower eigenvalues are shown in Fig. 2.3 in terms of ply orientation angle $\theta$.

The buckling mode with the lowest eigenvalue is shifted between the orientation angle $58^\circ$ and $59^\circ$ from the first mode of $m+n = \text{even}$ to the first mode of $m+n = \text{odd}$. Then, the order of the eigen mode is shifted from the first mode of $m+n = \text{even}$ to the second mode of $m+n = \text{even}$ between the orientation angle $67^\circ$ and $68^\circ$. Here, the order of eigenvalues is determined at $\theta = 45^\circ$ where the buckling load reaches the maximum value.
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Figure 2.3: Lower eigenvalues of a square plate $[(+\theta/ -\theta)]_S$ subject to uniaxial compression load.

![Lower eigenvalue plot](image)

Figure 2.4: Sensitivity of the buckling load and the second lowest mode with respect to $E_1$ of the outer-most ply.

![Sensitivity plot](image)

The sensitivities of the buckling load and the second lowest mode with respect to Young's modulus in the fiber direction $E_1$ of the outer-most ply are shown in Fig. 2.4. Corresponding to the buckling mode shift between $58^\circ$ and $59^\circ$, the sensitivities are also shifted. Since the modes are obtained from different eigenvalue problems, $m + n = \text{even}$ and $m + n = \text{odd}$, the corresponding eigen modes are independent of each other. Therefore, the sensitivity can be obtained from the method for a single eigenvalue.

On the other hand, the sensitivity of the second mode changes discontinuously between $67^\circ$ and $68^\circ$ where the mode is shifted from the first mode of $m + n = \text{even}$ and the second mode of $m + n = \text{even}$. Since both modes belong to the same eigenvalue problem, the sensitivity should be calculated from the sensitivity analysis for a repeated eigenvalue at the shifted point.
2.5.2 Sensitivity with Respect to Orientation Angles

The sensitivity of the buckling load with respect to ply orientation angle is shown in Fig. 2.5, where "4th ply" corresponds to the outer-most ply and "1st ply" to the inner-most ply which is the closest to the mid-plane of the plate.

The buckling sensitivity becomes discontinuous at the point where the buckling mode is shifted, as illustrated in Fig. 2.4. In order to compare the effect of the ply location, the sensitivities with respect to the absolute value of the orientation angle are given. That is, for the first and the third plies which have negative orientation angles, the values in Fig. 2.5 are $\partial \lambda / \partial (-\theta)$.

Between around $30^\circ$ and $58^\circ$, the sensitivities with respect to angles of the fourth and the third plies take almost the same value. It means that the effect of variation of the orientation angle of the third ply on the buckling load is almost equal to that of the outer-most ply. Since a buckling load is a function in terms of the bending stiffness, the effect of the outer ply is known to be larger than that of the inner ply, in general. However, the both plies have almost the same sensitivity. This is mainly due to the effect of the bending-torsion coupling terms $D_{16}$ and $D_{26}$, as discussed later.

2.5.3 Sensitivity with Respect to Material Constants

The effect of the variations of the material constants on the buckling load is determined by relative sensitivity which is a product of the material constants $x_i$ and the derivative with respect to itself ($\partial \lambda / \partial x_i$); $(\partial \lambda / \partial x_i) \times x_i$. The relative sensitivities are shown in Figs. 2.6(a) through 2.6(d) for each material constant. The sensitivity always takes positive values. Because the buckling load is a monotonously increasing function in terms of material constants.

Comparing the sensitivities, the effect of variation of $E_1$ on the buckling load is relatively
large. On the other hand, the effects of variations of \( E_2, E_4, \) and \( \nu_1 \) are small.

Focusing attention on the sensitivity with respect to \( E_1 \) in Fig. 2.6(a), the effect of the third ply is almost the same as that of the outer-most ply between 59° and 64°. This is due to the effect of the coupling terms as well as the sensitivity with respect to orientation angles (see Fig. 2.5).

The tendency also appears in case of shear buckling \( (N_x = N_y = 0, N_{xy} = 1) \), as shown in Fig. 2.7. The sensitivity of the third ply is larger than that of the fourth ply in almost all orientation angles. Especially, the sensitivity is about four times larger around 45°.

### 2.5.4 Effect of Bending Stiffness

As shown above, the sensitivity of the inner ply property takes larger value than that of the outer ply property in some cases, although the bending stiffness is a cubic function of the distance from the mid-plane to the ply location.
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Figure 2.6(c): Buckling load sensitivity with respect to $E_6$.

Figure 2.6(d): Buckling load sensitivity with respect to $\nu_1$.

Figure 2.7: Shear buckling load sensitivity with respect to $E_1$. 
Figure 2.8(a): Buckling load sensitivity with respect to \( D_{ij} \) in case of ignoring coupling terms \( D_{16} \) and \( D_{26} \).

Figure 2.8(b): Buckling load sensitivity with respect to \( E_1 \) in case of ignoring coupling terms \( D_{16} \) and \( D_{26} \).

In this subsection, the effect of each element of bending stiffness on the buckling load is investigated in both cases with and without the coupling terms.

**Without Coupling Terms**

The buckling equation of an orthotropic plate without the coupling terms \( D_{16} \) and \( D_{26} \) is shown in Eq. (2.26). As it is a linear equation in term of bending stiffness, the buckling sensitivities with respect to \( D_{ij} \) take constant values in the region of the same buckling half number \((m, n)\):

\[
\begin{align*}
\frac{\partial \lambda}{\partial D_{11}} &= \frac{\pi^2 m^4}{a^2(m^2 N_x + n^2 R^2 N_y)^{\frac{1}{2}}}, & \frac{\partial \lambda}{\partial D_{22}} &= \frac{\pi^2 n^4 R^4}{a^2(m^2 N_x + n^2 R^2 N_y)^{\frac{1}{2}}}, \\
\frac{\partial \lambda}{\partial D_{12}} &= \frac{2\pi^2 m^2 n^2 R^2}{a^2(m^2 N_x + n^2 R^2 N_y)^{\frac{1}{2}}}, & \frac{\partial \lambda}{\partial D_{66}} &= \frac{4\pi^2 m^2 n^2 R^2}{a^2(m^2 N_x + n^2 R^2 N_y)^{\frac{1}{2}}}
\end{align*}
\]  (2.56)
2.5. NUMERICAL CALCULATIONS

Figure 2.9: Buckling load sensitivity with respect to $D_{ij}$ in case of considering coupling terms $D_{16}$ and $D_{26}$.

Buckling sensitivities with respect to the $D_{ij}$ under the same load condition as the previous sections are illustrated in Fig. 2.8(a). The sensitivity is shown to be discontinuous at the place where the buckling mode is shifted. However, the sensitivity is constant in the same mode.

The sensitivity of the bending stiffness $D_{ij}$ with respect to material constants; $E_1$, $E_2$, $E_6$ and $\nu_1$ are proportional to $t^3 - (t - 1)^3$ as shown in Eq. (2.43). When the coupling term is ignored, the buckling sensitivities with respect to the material constants are also proportional to $t^3 - (t - 1)^3$. For example, the sensitivity with respect to $E_1$ is shown in Fig. 2.8(b). Comparing the sensitivity with respect to $E_1$ of the fourth ply, the sensitivity of the third ply is $(3^3 - 2^3)/(4^3 - 3^3) = 0.51$ times smaller. Similarly, that of the second ply is $(2^3 - 1^3)/(4^3 - 3^3) = 0.19$ times, and that of the inner-most ply is $1/(4^3 - 3^3) = 0.027$ times. The sensitivities with respect to other material constants or ply orientation angles behave similarly.

This tendency is completely different from the buckling sensitivity with coupling terms as discussed previously (see Figs. 2.5, 2.6(a) through 2.6(d)).

With Coupling Terms

The sensitivity with respect to the bending stiffness in case of considering coupling terms is shown in Fig. 2.9. $\partial \lambda / \partial D_{11}$, $\partial \lambda / \partial D_{22}$, $\partial \lambda / \partial D_{12}$ and $\partial \lambda / \partial D_{66}$ are not constant in terms of the ply orientation angles, though the changes are very small. On the other hand, changes of the sensitivity with respect to the coupling terms, $\partial \lambda / \partial D_{16}$ and $\partial \lambda / \partial D_{26}$ are relatively much larger. Also, the sensitivities take negative values.

The effect of each ply on the bending stiffness $D_{ij}$ is always proportional to cubic of the distance from the mid-plane to the ply location. Therefore, the effect of the outer-most ply is always dominant. When the coupling term is ignored, the effect of the bending stiffness
on the buckling load is constant as long as it belongs to the same buckling mode. Therefore, the effect of each ply on the buckling load is also proportional to cubic of the distance from the mid-plane to the ply location.

On the other hand, when the coupling terms are considered, the buckling sensitivity is much different from the ignored case. Especially, the sensitivities with respect to the coupling terms take negative values whose absolute values are relatively large. In some cases, the coupling terms $D_{16}$ and $D_{26}$ will dominate the effect of other terms. Therefore, the buckling sensitivity of the inner ply parameter takes larger values than that of the outer ply.

2.6 Summary

In this chapter, the buckling analysis for an unsymmetric laminated plate is introduced. Even if the mean laminate sequence is symmetric, the actual laminate configuration is unsymmetric due to variations of the material constants or the ply orientation angles of layers. Therefore, the unsymmetric modeling is required to evaluate the buckling load. The modeling will be used in the reliability analysis.

Then, the sensitivity analysis of the buckling load of the simply supported laminated plate with respect to the material constants and the ply orientation angle of each ply as well as the applied load is formulated. The sensitivity is utilized to investigate effects of variations of the parameters on variation of the buckling load. The sensitivity is also used in reliability analysis for evaluating the searching direction of the reliability.

From numerical calculations, the following conclusions are remarked.

1. Sensitivity of each eigenvalue is changed continuously. However, due to the buckling mode shift, the buckling load sensitivity is changed discontinuously at the buckling mode shifted point.

2. Buckling load sensitivity with respect to Young's modulus in the fiber orientation and the ply orientation angle have larger values. It means that the variations of the parameters have large effects on the variations of the buckling load.

3. The effect of the outer ply on the buckling load is known to be generally larger than that of the inner ply. Commonly, the outer ply has larger sensitivity than the inner ply. However, depending on the laminate configurations and applied loading conditions, the sensitivity of the inner ply becomes larger due to the existence of the bending-torsion coupling terms $D_{16}$ and $D_{26}$. 
Chapter 3

RELIABILITY ANALYSIS

3.1 Introduction

This chapter is devoted to the reliability analysis of a composite laminated plate subject to buckling, where the material constants and the ply orientation angle of each ply have some probabilistic variations as well as applied loads.

First, a modeling of the buckling failure is proposed. As described in Chapter 2, laminate configuration will be unsymmetric due to variations of random variables, even though the mean laminate is assumed to be symmetric. Therefore, the unsymmetric stacking construction should be considered in the reliability analysis. Moreover, since buckling mode may be shifted due to variations of random vectors, it is not sufficient to only consider the buckling mode at the mean design point. The buckling load factor which denotes the ratio of the buckling load to the applied load is obtained as a minimum eigenvalue. In order not to buckle the plate under the applied load condition, the minimum eigenvalue should be larger than unity. Hence, all of the eigenvalues should be larger than unity. Therefore, the buckling failure is modeled as a series system consisting of several eigen modes.

The mode reliability is evaluated by the FORM which can be formulated as a nonlinear programming problem. The limit state function of each eigen mode is defined as the condition where the eigenvalue should be larger than unity. Following the evaluation of the mode reliabilities, the system reliability is approximated by Ditlevsen’s bounds.

In eigenvalue analysis, eigen solutions are ordered by eigenvalue magnitude. Since mode crossings may occur by variations of random variables, the mode cannot be identified by the eigenvalue order during the mode reliability searching process. Therefore, the mode tracking strategy is required to keep track of the intended mode.

Through numerical calculations, the proposed buckling failure model is shown to have a sufficient accuracy in comparison with Monte Carlo simulation. Then, it is shown that the Young’s modulus in the fiber direction and the ply orientation angle of each ply as well as applied load have dominant effect on the reliability. Finally, effects of variations of applied loads are illustrated by several designs with different orientation angles.
3.2 First Order Reliability Method

The uncertain quantities are modeled as a vector of \( n \) physical stochastic variables \( \mathbf{X} = (X_1, \cdots, X_n)^T \). A failure element modeling a specific potential failure mode in the structure is described by a limit state function \( M(\mathbf{x}) \). For realizations \( \mathbf{x} \) of \( \mathbf{X} \), the limit state function divides the \( \mathbf{X} \)-space into two regions: failure region, \( M(\mathbf{x}) \leq 0 \), and safe region \( M(\mathbf{x}) > 0 \). The reliability \( R \) of the element can be written as:

\[
R = 1 - P_f = 1 - \int_{M(\mathbf{x}) \leq 0} f_X(\mathbf{x}) \, d\mathbf{x}
\]  

(3.1)

where \( P_f = \int_{M(\mathbf{x}) \leq 0} f_X(\mathbf{x}) \, d\mathbf{x} \) is the failure probability, and \( f_X(\mathbf{x}) \) is the joint probability density function of random variables \( \mathbf{X} \).

In the FORM, the approximation of Eq. (3.1) is obtained by introducing a \( \mathbf{U} \)-space of standardized independent and normally distributed variables by the transformation \( \mathbf{X} = \mathbf{T}(\mathbf{U}) \).

In the \( \mathbf{U} \)-space, the reliability index \( \beta^e \) is defined as the minimum distance from the origin. In order to obtain the reliability index, an iteration algorithm is proposed by Rackwitz and Fiessler, (1978). Recently, the modified algorithm using an adaptive nonlinear approximation with introduction of intermediate variables is proposed in order to achieve rapid convergence by Wang and Grandhi (1994). In this thesis, the reliability is obtained through a nonlinear programming problem which is recommended by Shinozuka (1983).

The reliability index is formulated as the solution of the following optimization problem:

\[
\text{Minimize} \quad : \quad \beta^e = \sqrt{\mathbf{u}^T \mathbf{u}}
\]

subject to \( g(\mathbf{u}) = M(\mathbf{T}(\mathbf{u})) = 0 \)  

(3.2)

The solution of the problem \( \mathbf{u}^* \) is called the design point, most probable point, or \( \beta \) point, where \( \beta^e = \sqrt{\mathbf{u}_{*}^T \mathbf{u}_{*}} \). Then, the limit state function \( g(\mathbf{U}) \) is linearized around the design point:

\[
g(\mathbf{U}) \approx -\alpha^T \mathbf{U} + \beta^e
\]  

(3.3)

where \( \alpha \) is the normalized vector of negative direction cosine of \( \mathbf{u}_{*} \):

\[
\alpha = \frac{\mathbf{u}_{*}}{\beta^e} = \frac{-\nabla_u g(\mathbf{u}_{*})}{| -\nabla_u g(\mathbf{u}_{*})|}
\]  

(3.4)

where it is implied that the first order derivatives of the limit state function exist. The failure probability \( P_e \) of the failure mode is then approximately determined from \( P_e = \Phi_1(-\beta^e) \), where \( \Phi_1(\cdot) \) is the one dimensional standard normal distribution function defined as follows:

\[
\Phi_1(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\tau^2}{2}\right) \, d\tau
\]  

(3.5)
3.3 Series System

A number of failure elements are assigned to the structural system. If the whole structural system is modeled as a series system of $m$ failure modes, i.e., failure of the system is defined as failure of any one element, the failure probability $P_f$ can be written as a probability of unions:

$$P_f = P\left(\bigcup_{i=1}^{m}\{g_i(U) \leq 0\}\right)$$

(3.6)

and the generalized system reliability index $\beta^s$ can be estimated as

$$\beta^s = -\Phi_1^{-1}(1 - \Phi_m(\beta^e, \rho))$$

(3.7)

where $\Phi_m(\cdot)$ is the $m$-dimensional standardized normal density function, $\beta^e$ is the vector of reliability indices of the individual failure modes $\beta^e = (\beta^e_1, \ldots, \beta^e_m)^T$ and $\rho$ is the corresponding matrix of correlation coefficients defined as

$$\rho = A^T A = [\alpha_1, \ldots, \alpha_m]^T [\alpha_1, \ldots, \alpha_m]$$

(3.8)

where $A = [\alpha_1, \ldots, \alpha_m]$ and $\alpha_i$ is the normalized vector of negative direction cosine of design point of the $i$-th failure mode.

The multi-dimensional normal distribution function $\Phi_m$ is defined by

$$\Phi_m(\beta, \rho) = \int_{-\infty}^{\beta_1} \cdots \int_{-\infty}^{\beta_m} \phi_m(x, \rho) \, dx_1 \cdots dx_m$$

(3.9)

where $\phi_m$ is the $m$-dimensional normal density function

$$\phi_m(x, \rho) = \frac{1}{(2\pi)^{m/2}||\rho||^{1/2}} \exp\left[-\frac{1}{2} x^T \rho^{-1} x\right]$$

(3.10)

However, the multi-dimensional normal distribution with correlation is difficult to calculate. Therefore, the probability is estimated by an approximation. In this thesis, Ditlevsen’s bounds are used, Ditlevsen (1979):

$$P_L \leq P_f \leq P_U$$

$$P_L = P_1 + \sum_{i=2}^{m} \max\left[P_i - \sum_{j=1}^{i-1} P_{ij}, 0\right]$$

(3.11)

$$P_U = \sum_{i=1}^{m} P_i - \sum_{i=2}^{m} \max_{j<i} P_{ij}$$

where $P_{ij}$ means the joint failure probability of events $i$ and $j$. In Fig. 3.1, the region of $E_i \cap E_j$ corresponds to the joint failure region. The joint probability is defined as follows:

$$P_{ij} = \Phi_2(-\beta_i, -\beta_j; \rho_{ij}) = \int_{-\infty}^{-\beta_1} \int_{-\infty}^{-\beta_2} \frac{1}{2\pi \sqrt{1-\rho_{ij}^2}} \exp\left\{-\frac{x^2 + y^2 - 2\rho_{ij}xy}{2(1-\rho_{ij}^2)}\right\} \, dx \, dy$$

(3.12)
$\Phi_2(\cdot)$ means the two dimensional standardized normal distribution function and $\rho_{ij}$ is the correlation coefficient between two modes:

$$\rho_{ij} = \frac{u_i^T u_j}{|u_i||u_j|} = \alpha_i^T \alpha_j = \cos \mu$$

(3.13)

where $\mu$ is the angle between the vector $u_i$ and $u_j$ as shown in Fig. 3.1.

The generalized reliability indices corresponding to the lower and the upper bounds are evaluated as follows:

$$\beta_U \leq \beta^* \leq \beta_L$$
$$\beta_L = -\Phi^{-1}(P_L)$$
$$\beta_U = -\Phi^{-1}(P_U)$$

(3.14)

### 3.4 Reliability of Buckling Failure

#### 3.4.1 Modeling of Buckling Failure

In practical structural applications, a symmetric laminated plate is mainly used to avoid a bending-extension coupling which makes the structural behavior complicate. However, the actual stacking sequence of a laminated plate will be unsymmetric, even though the mean stacking sequence is assumed to be symmetric due to the random variations of material properties and orientation angle of each ply. Consequently, it is required to consider an unsymmetric configuration for reliability analysis.

The buckling failure occurs when the eigenvalue is less than unity. In other words, all the eigenvalues should be larger than unity in order not to buckle the plate. Therefore, it is suggested that the buckling failure event can be considered as a series system consisting of eigen modes.
3.4. RELIABILITY OF BUCKLING FAILURE

There exist $M \times N$ failure modes, and all the eigen modes may have some probabilities of occurrence. However, some have high probabilities and others have relatively low ones. It is estimated that an eigen mode of a lower eigenvalue will be more critical than that of a higher eigenvalue. Because the lower mode is always closer to the failure surface than the higher mode. In this problem, $N_f$ eigen modes taken from the minimum eigenvalue to the $N_f$-th in an ascending order at the mean value point are selected as dominant failure modes.

The limit state functions of the series system are written as follows:

$$g_i(u) = \lambda_i(u) - 1 \geq 0, \quad (i = 1, \cdots, N_f) \quad (3.15)$$

where $\lambda_i$ is the eigenvalue corresponding to the $i$-th failure mode, $N_f$ is the number of failure modes, and $u$ denotes the standardized random vector. The material property and the orientation angle of individual layers as well as the applied loads are treated as independent random variables.

### 3.4.2 Illustrative Example

The importance of modeling a buckling failure as the series system is illustrated through a simple example.

Fig. 3.2 shows the buckling load contour plot in terms of axial compression loads of $x$ and $y$ directions for the $[(+49^\circ/-49^\circ)_{2s}]_S$ plate whose aspect ratio is 1.5. The standardized load components are varying in $200$(GPa) $\leq N_x^* \leq 400$(GPa) and $-100$(GPa) $\leq N_y^* \leq 100$(GPa).
100(GPa). Note that the positive value indicates the compression load and the negative value indicates the tensile load. The applied load is standardized with respect to the plate dimension, which will be given by Eq. (3.19) in Section 3.6, later.

The plotted contour lines indicate \( \lambda = 0.9, 1.0, 1.1 \) and 1.21. The line of \( \lambda = 1.0 \) will be a limit state curve. The gentle slope line in the upper left side of Fig. 3.2 corresponds to the first mode of \( m + n = \text{odd} \) and the steep slope line in the right side to the first mode of \( m + n = \text{even} \). Two modes are shifted at the cross point of the two lines.

Here, consider the case where the applied load \( N_{x}^* \) and \( N_{y}^* \) have variations whose mean values are \( \bar{N}_{x} = 300(\text{GPa}) \) and \( \bar{N}_{y} = 0(\text{GPa}) \) and their standard deviations are the same. The mean value point corresponds to the center point in Fig. 3.2. When the random variables are translated into the \( U \)-space, the central point is moved to the origin. Since both variables are assumed to have the same standard deviation, the contour lines maintain the same forms in the \( U \)-space without considering the coordinate values in the \( U \)-space which depend on the standard deviations.

When the mode is not considered in the reliability analysis, the limit state function is defined as follows:

\[
g(U) = \min \left\{ \lambda_i(U) \right\} - 1
\]

If the reliability searching process is started from the origin, the process will be converged to point B in Fig. 3.2. Because the searching direction which is perpendicular to the contour line through the origin goes directly toward the point B. However, the global optimal point is A, as the circle in Fig. 3.2 indicates. The mode corresponding to the point A is different from the buckling modes of the mean value point and the point B.

In order to obtain the global optimal point A, the reliability searching should be performed for the first mode of \( m + n = \text{even} \). However, the mode cannot be identified in advance. Therefore, the reliability analysis should be performed for several eigen modes. That is, the series system modeling is required.

### 3.5 Mode Tracking Strategy

#### 3.5.1 Simple Eigenvalues

In order to evaluate a mode reliability, the corresponding limit state function should be identified in each iteration. The failure modes are defined as potential eigen modes whose corresponding eigenvalues are low at the mean value point. In numerical analysis, the eigenvalues and the eigenvectors are ordered by eigenvalue magnitude. Then, the specific mode must be referenced by a number. However, during the iteration in the reliability analysis, mode crossing may occur due to the random vector perturbations. If these crossings are not tracked, the limit state function may be evaluated by using different modes from the intended one. This causes problems from two aspects; the reliability will not reflect the
intended mode and the numerical searching will not converge. Therefore, a mode tracking strategy is required to keep track of the intended mode.

The strategy has been developed for the experimental modal identification or the modal tuning, Ting et al. (1995) and Gibson (1992), and the structural dynamics optimization problem, Eldred et al. (1995). The modal tuning can be formulated as mathematical programming to minimize discrepancies between the experimental data and the analytical response in terms of these parameters.

The both strategies proposed by Ting et al. (1995) and Gibson (1992) are based on an eigenvector orthonormality. At each iteration, the following eigenvector orthogonality check is performed:

\[ C_{ij} = \phi_i^{(k-1)T} K_G^{(k)} \phi_j^{(k)} \]  

where suffixes \((k)\) and \((k-1)\) denote the current and the previous iterations, respectively. If the \(i\)-th mode at the previous iteration corresponds to the current \(j\)-th mode, \(C_{ij}\) is nearly equal to one. Otherwise, \(C_{ij}\) is nearly equal to zero. Using this strategy, the corresponding mode number is always monitored.

On the other hand, the strategy by Eldred et al. (1995) is based on the higher order eigenpair perturbations. Comparing with the cross orthogonality check method, the method is more robust but requires more computational time.

Therefore, the cross orthogonality check is adopted here. In order to achieve a good accuracy of the mode tracking, the method is modified as follows. When the eigenvector change is large due to the large design change, the cross check may fail. In this study, the strategy is modified in order to improve the accuracy. Replacing the previous eigenvector by the first order Taylor expansion around the previous position to the current position, the accuracy will be improved. However, the derivatives of the eigenvectors require high computational costs. The \(\beta\)-searching process is performed around the mean value point \((u = 0)\) for almost all cases. Moreover, the design change in the reliability analysis is relatively smaller than that in the structural optimization problem. Therefore, the cross check is performed between the current eigenvector and the Taylor expanded vector around the mean eigenvector of the current position. Thus,

\[ C_{ij} = \phi_i^{\text{opp}}(u) K_G^{(k)} \phi_j^{(k)} \]  

\[ \phi_i^{\text{opp}}(u) = \phi_i^{(0)} + \nabla \phi_i^{(0)} T u \]  

where \(\phi_i^{(0)}\) and \(\nabla \phi_i^{(0)}\) are the mean eigenvector and the eigenvector derivative with respect to the random vector at the mean value point, respectively.

The modification requires the derivatives of eigenvectors, but it is only once before starting the reliability searching. The method is described in Appendix B.
### Table 3.1: Variations of material constants and ply angle.

<table>
<thead>
<tr>
<th></th>
<th>$E_1$ (GPa)</th>
<th>$E_2$ (GPa)</th>
<th>$E_6$ (GPa)</th>
<th>$\nu_1$</th>
<th>$\theta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>181.0</td>
<td>10.3</td>
<td>7.17</td>
<td>0.28</td>
<td>—</td>
</tr>
<tr>
<td>Cov†</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>SD†: 5°</td>
</tr>
</tbody>
</table>

†: coefficient of variation, ††: standard deviation.

### Table 3.2: Applied load conditions.

<table>
<thead>
<tr>
<th></th>
<th>$N_x^*$ (GPa)</th>
<th>$N_y^*$ (GPa)</th>
<th>$N_{xy}^*$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean value</td>
<td>300.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>SD†</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>COV = 0.00</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>COV = 0.02</td>
<td>6.0</td>
<td>6.0</td>
<td>6.0</td>
</tr>
<tr>
<td>COV = 0.05</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0</td>
</tr>
</tbody>
</table>

††: SD = standard deviation.

### 3.5.2 Repeated Eigenvalues

If the $j$-th eigenvalue at current design is repeated, the corresponding eigenvector $\phi_j^{(k)}$ may fail in the orthogonality check. That is, $C_{ij}$ may not have any dominant element. In such a case, the obtained eigenvectors are not continuous with the previous eigenvectors. Since any linear combinations of eigenvectors corresponding to the repeated eigenvalues will satisfy the eigenvalue equation, such eigenvectors may be obtained.

Therefore, the eigenvectors should be translated into the vector which is continuous with the eigenvector at the mean value point. For this purpose, the derivatives for the repeated eigenvalues are utilized.

### 3.6 Numerical Calculations

In this section, the general conclusions of the reliability analysis of a laminated plate subject to buckling load are illustrated by some numerical examples.

Numerical calculations are performed for two types of simply-supported 8-ply angle-ply laminated plates $[\pm\theta/\pm\theta]_s$ made up with T300/5208 (graphite/epoxy) subject to uniaxial compression load with and without variations. One is for a square plate (aspect ratio $R = 1.0$) and the other is for a rectangular (aspect ratio $R = 1.5$). The material constants and the ply orientation angle of each layer are assumed to have independent normal distribution. The means and the coefficients of variation are listed in Table 3.1.

The mean values are assumed to be about 80% of the deterministic maximum buckling load in terms of orientation angle, and three types of variations are considered as listed in Table 3.2. The applied load is standardized with respect to plate dimensions as follows,
3.6. NUMERICAL CALCULATIONS

Table 3.3: Mean buckling load and reliability of \([(+55^\circ/ - 55^\circ)_2]_s\) plate subject to uniaxial compression load with variation, (COV= 0.05).

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Mean eigenvalue</th>
<th>Mode reliability</th>
<th>System reliability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\lambda_{\text{even}}^i)</td>
<td>(\lambda_{\text{odd}}^i)</td>
<td>(\beta_{\text{even}}^i)</td>
</tr>
<tr>
<td>1</td>
<td>1.2359</td>
<td>1.1051</td>
<td>2.3489</td>
</tr>
<tr>
<td>2</td>
<td>1.4764</td>
<td>1.5207</td>
<td>3.6647</td>
</tr>
<tr>
<td>3</td>
<td>1.9103</td>
<td>2.3952</td>
<td>5.8790</td>
</tr>
</tbody>
</table>

\[ N^* = \frac{12a^2}{\pi^2h^3R^2}N \]  \hspace{1cm} (3.19)

where \(a\), \(b\) and \(h\) are a plate length, width and thickness, respectively.

Note that in the buckling analysis, the maximum number of the double sine series of the assumed displacement in Eq. (2.17) is set to \(M = N = 10\). That is, the matrix size of eigenvalue equations are 50 x 50 for both cases of \(m + n = \text{even}\) and \(m + n = \text{odd}\).

3.6.1 Mode Reliability and Mode Tracking

The mode tracking strategy is shown to be effective to capture the intended mode in the reliability analysis. One example is given from the mode reliability analysis for the second mode of \(m + n = \text{even}\) of \([(+55^\circ/ - 55^\circ)_2]_s\) plate whose aspect ratio is \(R = 1.5\) subject to uniaxial compression load with variations (COV=0.05). The mean buckling load and the result of the reliability analysis are listed in Table 3.3.

The iteration history is shown in Fig. 3.3. At the second iteration, the tracked mode is changed from the second mode to the first mode. At the point, the cross orthogonality check of Eq. (3.17) results in \(C_{21} = 0.8717 \approx 1\) and \(C_{22} = 0.1967 \approx 0\). Consequently, the first mode is selected as the intended mode. If the strategy is not used, the reliability will
Figure 3.4(a): Mode shape contour plots of the 1st mode of \( m + n = \text{even} \) at the mean value point, \( \lambda_{\text{even}}^1 = 1.2359 \).

Figure 3.4(b): Mode shape contour plots of the 2nd mode of \( m + n = \text{even} \) which is an intended mode at the mean value point, \( \lambda_{\text{even}}^2 = 1.4764 \).

be evaluated on the first mode which will yield a wrong result. Finally at the design point, the reliability is evaluated for the first mode which corresponds to the second mode at the mean value point.

To make sure whether the first mode at the design point corresponds to the intended mode (the second mode of \( m + n = \text{even} \) at the mean value point), the mode deformation shapes are compared. Figs. 3.4(a) and 3.4(b) show the mode shape contour plots of the first and the second modes of \( m + n = \text{even} \) at the mean value point, respectively. On the other hand, Figs. 3.5(a) and 3.5(b) illustrate the contour plots at the design point. The first mode at the design point (Fig. 3.5(b)) corresponds to the second mode at the mean value point which is the intended mode (Fig. 3.4(a)). It means that the mode tracking strategy certainly keeps track of the intended mode during the mode reliability analysis.

When the mode tracking strategy is not used, the obtained mode reliability is \( \beta_{\text{even}}^2 = \)
3.6 NUMERICAL CALCULATIONS

Figure 3.5(a): Mode shape contour plots at the design point of the tracked mode which is the 1st mode of \( m + n = \text{even} \), \( \lambda_{\text{even}}^1 = 1.0000 \).

Figure 3.5(b): Mode shape contour plots at the design point of the 2nd mode of \( m + n = \text{even} \), \( \lambda_{\text{even}}^2 = 1.1241 \).

4.0689 which overestimates the reliability. The mode shape is also different as shown in Fig. 3.6.

3.6.2 Comparison with Monte Carlo Simulation

The estimated reliability by the FORM is shown to be suitable for the purpose of this thesis in comparison with Monte Carlo simulation.

Reliability indices are compared in Fig. 3.7(a) for aspect ratio = 1.0 and Fig. 3.7(b) for aspect ratio = 1.5, subject to uniaxial compression load with some variations (COV = 0.05). In Monte Carlo simulation, the reliability is obtained through \( 1 \times 10^5 \) times of buckling analyses, where the number is required to reduce the estimated coefficient of variation \( \tilde{\text{Cov}} \) of the failure probability.

\( \tilde{\text{Cov}} \) lies between 0.6 and 3.7 % after \( 1 \times 10^5 \) analyses, as listed in Table 3.4. After
Figure 3.6: Mode shape contour plots of 2nd mode of $m + n = \text{even}$ at the design point without mode tracking.

Figure 3.7(a): Reliability index in comparison with Monte Carlo simulation, aspect ratio $R = 1.0$.

Figure 3.7(b): Reliability index in comparison with Monte Carlo simulation, aspect ratio $R = 1.5$. 
Table 3.4: Estimated coefficient of variation of the failure probability in Monte Carlo simulation.

(a) Aspect ratio, $R = 1.0$.

<table>
<thead>
<tr>
<th>Ply angle (deg)</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cov (%)</td>
<td>0.71</td>
<td>1.43</td>
<td>2.56</td>
<td>3.21</td>
<td>2.56</td>
<td>1.36</td>
</tr>
</tbody>
</table>

(b) Aspect ratio, $R = 1.5$.

<table>
<thead>
<tr>
<th>Ply angle (deg)</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>47</th>
<th>50</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cov (%)</td>
<td>0.60</td>
<td>1.34</td>
<td>3.46</td>
<td>3.70</td>
<td>2.49</td>
<td>0.74</td>
</tr>
</tbody>
</table>

Figure 3.8: Contour plot of buckling load in terms of orientation angles of both outer-most plies.

estimated probabilities with one standard deviation $\hat{P}_f \pm \sigma$ are translated into reliability indices, regions of the estimated reliabilities with one standard deviation are shown in Figs. 3.7(a) and 3.7(b) by vertical bars with caps. But the regions are so small that only caps can be seen in some examples.

It is found that the FORM does not have a good accuracy and overestimates the reliability on most points for both cases. However, the changes of the reliability with respect to the orientation angle are quite similar for both methods. Therefore, the FORM is useful to compare the reliability between different designs qualitatively. It means that the FORM is suitable for the reliability-based optimization which compares the reliability between different designs.

The difference between the reliability by the FORM and that by Monte Carlo simulation
is mainly caused by an approximation error of the linearization of a limit state function in the FORM. The buckling load is strongly nonlinear in terms of the orientation angles, especially of the outer ply. To investigate how the ply orientation angle affects the buckling load, the buckling load contour with respect to the outer ply angles is plotted in Fig. 3.8. Orientation angles of the outer-most plies of \[ (\pm 35^\circ / - 35^\circ)_{2s} \] square plate (aspect ratio; \( R = 1.0 \)) subject to uniaxial compression load are varied \( \pm 10^\circ \). The abscissa axis is the ply orientation angle of the bottom layer and the ordinate axis is that of the top layer. The center point corresponds to \([(+35^\circ / - 35^\circ)_{2s}]\). The ply orientation angles of the remaining six layers are kept constant. Laminate construction is symmetric along the line whose slope is \( 45^\circ \) through the origin. For example, the top right corner corresponds to \([+45^\circ / -35^\circ]_{2s} \) and the bottom left corner to \([+25^\circ / -35^\circ / +35^\circ / -35^\circ]_{2s} \). On the other hand, the laminate becomes unsymmetric off the \( 45^\circ \) line. The top left and the bottom right corner correspond to the same unsymmetric laminate, \([+25^\circ / -35^\circ / +35^\circ / -35^\circ]_{2s} / +45^\circ \). The limit state surface is realized on the curve of \( \lambda = 1.0 \) on this plane, which will be of the similar form as other contour lines. From a view of the mean value point, the limit state surface will have a concave form also in the \( U \)-space. Therefore, the FORM which is based on the linearization of the limit state function along a design point will underestimate the failure probability, and hence overestimate the reliability.

### 3.6.3 Unsymmetric Model

In many practical applications, bending-extension and shear-extension couplings are undesirable. Consequently, most laminates in use today are symmetric and balanced to eliminate these couplings. Besides, a balanced symmetric laminate is much easier to analyze. The symmetric model requires only a half side of layers under the assumption that layers in another side which locate in the same position with respect to the mid-plane of a laminate have the same property. Therefore, most of optimization works have been limited to balanced symmetric laminates, e.g., Haftka and Gurdal (1992). Moreover, when the orientation angles are used as design variables, the required number of design variables can be reduced to one-fourth of the number of layers.

However, in the reliability analysis of the composite plates where the layer properties are used as random variables, the symmetric laminate model may yield wrong results. Consider the case when laminate properties such as layer material constants or ply orientation angles, have some variations and are used as random variables. Even if the mean value and the variation of the property in a layer are the same as those in the corresponding layer with respect to the mid-plane, the realized properties will be different except for the case where the properties have a positive perfect correlation. Therefore, the unsymmetric model is required in the reliability analysis. Here, the importance of the unsymmetric model is illustrated through numerical examples.
3.6. NUMERICAL CALCULATIONS

Figure 3.9: Reliability of symmetric model and unsymmetric model.

The reliability of both models are compared in Fig. 3.9 in the case of a square plate (aspect ratio, $R = 1.0$) subject to a uniaxial compression with no variation (COV=0.00). The abscissa axis is the mean orientation angle and the ordinate axis is the system reliability. $\beta_{\text{sym}}$ is the reliability obtained by the symmetric model, and $\beta_{\text{un}}$ is by the unsymmetric model. The figure shows that the symmetric model overestimates the reliability in the wide region of orientation angle. Especially, at $\theta = 44^\circ$, the reliability by the symmetric model is $\beta_{\text{sym}} = 7.464$, which is about twice larger than that by the unsymmetric model; $\beta_{\text{un}} = 4.066$.

3.6.4 Design Point

In the FORM, coordinates of the design point in the $U$-space give information about the effect of the variations of the design parameters on the buckling.

For the square plate of $\theta = 45^\circ$ subject to uniaxial compression load with no variations (COV= 0.00), the coordinates of design point in the first mode of $m + n = \text{even}$ which is the most dominant failure mode are listed in Table 3.5; (a) is the coordinate in the $U$-space and (b) is the corresponding coordinate in the $X$-space. The laminate construction of the design point is found to be unsymmetric.

The absolute values of coordinates of $U_{E_1}$ and $U_{\theta}$ in the outer layer take larger value. While, the value of $U_{E_2}$, $U_{E_6}$ and $U_{v_1}$ is small. From this point, the important factors of the buckling reliability are the variations of $E_1$ of outer plies and $\theta$.

The design point of the same plate in the case where the applied load has some variations (COV=0.05) is listed in Table 3.6. The variations of the $N_x^e$ and $N_y^e$ are the most effective on the variation of the buckling load. Moreover, the variations of $E_1$ and $\theta$ in the outer plies are also important.
Table 3.5: Coordinate of design point.
(a) Coordinate in the $U$-space, $\beta_1 = 4.230$.

<table>
<thead>
<tr>
<th>Ply</th>
<th>$U_{E_1}$</th>
<th>$U_{E_2}$</th>
<th>$U_{E_6}$</th>
<th>$U_{\nu_1}$</th>
<th>$U_\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper4</td>
<td>-0.444</td>
<td>-8.14E-2</td>
<td>-5.74E-2</td>
<td>-8.31E-3</td>
<td>2.488</td>
</tr>
<tr>
<td>Upper3</td>
<td>-0.536</td>
<td>-3.22E-2</td>
<td>-3.61E-2</td>
<td>-4.41E-3</td>
<td>-1.162</td>
</tr>
<tr>
<td>Upper2</td>
<td>-0.101</td>
<td>-1.56E-2</td>
<td>-4.11E-2</td>
<td>-1.32E-3</td>
<td>0.928</td>
</tr>
<tr>
<td>Upper1</td>
<td>-3.01E-2</td>
<td>-1.59E-3</td>
<td>-3.96E-2</td>
<td>-2.57E-4</td>
<td>-0.180</td>
</tr>
<tr>
<td>Lower1</td>
<td>-3.00E-2</td>
<td>-1.59E-3</td>
<td>-3.96E-2</td>
<td>-2.54E-4</td>
<td>0.179</td>
</tr>
<tr>
<td>Lower2</td>
<td>-0.101</td>
<td>-1.56E-2</td>
<td>-4.11E-2</td>
<td>-1.32E-3</td>
<td>-0.930</td>
</tr>
<tr>
<td>Lower3</td>
<td>-0.536</td>
<td>-3.22E-2</td>
<td>-3.61E-2</td>
<td>-4.38E-3</td>
<td>1.152</td>
</tr>
<tr>
<td>Lower4</td>
<td>-0.443</td>
<td>-8.13E-2</td>
<td>-5.77E-2</td>
<td>-8.34E-3</td>
<td>-2.493</td>
</tr>
</tbody>
</table>

(b) Laminate configuration.

<table>
<thead>
<tr>
<th>Ply</th>
<th>$E_1$ (GPa)</th>
<th>$E_2$ (GPa)</th>
<th>$E_6$ (GPa)</th>
<th>$\nu_1$</th>
<th>$\theta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper4</td>
<td>176.98</td>
<td>10.26</td>
<td>7.149</td>
<td>0.280</td>
<td>57.44</td>
</tr>
<tr>
<td>Upper3</td>
<td>176.15</td>
<td>10.28</td>
<td>7.157</td>
<td>0.280</td>
<td>-50.81</td>
</tr>
<tr>
<td>Upper2</td>
<td>180.09</td>
<td>10.29</td>
<td>7.155</td>
<td>0.280</td>
<td>49.64</td>
</tr>
<tr>
<td>Upper1</td>
<td>180.73</td>
<td>10.30</td>
<td>7.156</td>
<td>0.280</td>
<td>-45.90</td>
</tr>
<tr>
<td>Lower1</td>
<td>180.73</td>
<td>10.30</td>
<td>7.156</td>
<td>0.280</td>
<td>-44.10</td>
</tr>
<tr>
<td>Lower2</td>
<td>180.09</td>
<td>10.29</td>
<td>7.155</td>
<td>0.280</td>
<td>40.35</td>
</tr>
<tr>
<td>Lower3</td>
<td>176.15</td>
<td>10.28</td>
<td>7.157</td>
<td>0.280</td>
<td>-39.24</td>
</tr>
<tr>
<td>Lower4</td>
<td>176.99</td>
<td>10.26</td>
<td>7.149</td>
<td>0.280</td>
<td>32.54</td>
</tr>
</tbody>
</table>

Table 3.6: Design point in case of COV=0.05, $\beta_1 = 2.907$.

<table>
<thead>
<tr>
<th>Ply</th>
<th>$U_{E_1}$</th>
<th>$U_\theta$</th>
<th>$U_{load}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper4</td>
<td>-0.436</td>
<td>-0.776</td>
<td>$U_{N_x}$</td>
</tr>
<tr>
<td>Upper3</td>
<td>-0.409</td>
<td>0.298</td>
<td>$U_{N_y}$</td>
</tr>
<tr>
<td>Upper2</td>
<td>-8.41E-2</td>
<td>-0.280</td>
<td>$U_{N_{xy}}$</td>
</tr>
<tr>
<td>Upper1</td>
<td>-2.17E-2</td>
<td>4.91E-2</td>
<td></td>
</tr>
<tr>
<td>Lower1</td>
<td>-2.16E-2</td>
<td>-4.97E-2</td>
<td></td>
</tr>
<tr>
<td>Lower2</td>
<td>-8.41E-2</td>
<td>0.279</td>
<td></td>
</tr>
<tr>
<td>Lower3</td>
<td>-0.409</td>
<td>0.308</td>
<td></td>
</tr>
<tr>
<td>Lower4</td>
<td>-0.436</td>
<td>-0.773</td>
<td></td>
</tr>
</tbody>
</table>
3.6. NUMERICAL CALCULATIONS

3.6.5 Mode and System Reliabilities

The relation between the system reliability and the mode reliabilities are illustrated through two examples. They are the laminates of aspect ratio $R = 1.5$ subject to uniaxial compression load with and without variations; one is in the case of $COV = 0.00$ and the other is in the case of $COV = 0.05$.

The buckling reliability is approximated by Ditlevsen's upper bound of the series system consisting of potential eigen modes. From values of limit state functions at the mean value point design, it is considered that failure modes which have lower eigenvalues at the mean value point will be dominant modes, intuitively. Even if mode crossings occur, the mode is commonly shifted to the neighbour mode. The failure mode whose eigenvalue is much higher than the buckling mode can be ignored. Therefore, failure modes are limited to the eigen modes whose eigenvalues at the mean value point design are lower than 2.5 here. In this case, from the preliminary investigation, the maximum number is set to six and the minimum number is limited to two which are the first modes of $m + n = \text{even}$ and $m + n = \text{odd}$.

First, the deterministic lower eigenvalues are shown in Fig. 3.10. The buckling mode is shifted from $\lambda_{\text{even}}^1$ to $\lambda_{\text{odd}}^1$ around $42^\circ$ where the buckling load factor reaches the maximum and again from $\lambda_{\text{odd}}^2$ to $\lambda_{\text{even}}^2$ between $64^\circ$ and $65^\circ$. The first and the second modes of $m + n = \text{even}$ are shifted between $47^\circ$ and $48^\circ$, though mode identification number denoted by superscript is given by the ascending order. The third mode of $m + n = \text{even}$ is actually shifted around $68^\circ$ to higher eigen mode which is not shown here. When the orientation angle is smaller, eigen modes are much separated. On the other hand, as the angle becomes larger, the modes get close to each other.

Then, the mode reliabilities and the system reliability are illustrated in Fig. 3.11(a) for the load without variation; $COV = 0.00$, and Fig. 3.11(b) for the load with variation; $COV$
Figure 3.11(a): System and mode reliabilities under deterministic applied load; COV = 0.00.

Figure 3.11(b): System and mode reliabilities under variable applied load; COV = 0.05. Ditlevsen's lower bound is not indicated in the figures, because the difference from the upper bound $\beta_U$ is too small to identify in the figures. For both cases, when orientation angle is lower, the system reliability is dominated by only one failure mode; the first mode of $m + n = \text{even}$ and the next mode reliability is much higher. On the other hand, when the angle is larger, mode reliabilities get close to each other and the system reliability is dominated by more than two modes. The changes of mode reliabilities are similar to the changes of deterministic lower eigen modes at the mean value point design, as shown in Fig. 3.10.

Corresponding to the changes of mode reliabilities, the upper bound $\beta_U$ is almost equal to the mode reliability $\beta_{\text{even}}^1$ for the lower orientation angles. On the other hand, for higher orientation angles, the system reliability is dominated by more than two modes and is lower than the lowest mode reliability.

For both cases, two mode reliabilities $\beta_{\text{even}}^1$ and $\beta_{\text{even}}^2$ are shifted between $47^\circ$ and $48^\circ$
which corresponds to the shift of eigen modes, as shown in Fig. 3.10. For the deterministic load (COV=0.00), the two mode reliabilities are almost the same. On the other hand, for the case of variable load (COV=0.05), the mode reliabilities are much different from each other, though the deterministic eigenvalues of the mean value point design are equal to each other. This is attributed to the effect of the variation of applied loads.

Between 42° and 47° in the case of COV = 0.05, the lowest mode reliability is \( \beta_{\text{even}} \), though the lowest eigen mode is the first mode of \( m + n = \text{odd} \) in deterministic design, as shown in Fig. 3.10. This indicates that the lowest eigen mode at the mean value point design will not always correspond to the most critical failure mode. Therefore, more than two failure modes should be evaluated in the reliability analysis.

In comparison with these figures, the system reliability is known to be generally lower when the load variation is considered. The reliability maximization design is changed from around 42° to around 48°.

### 3.6.6 Effect of Load Variations

Reliability indices under three cases of load variations are shown in Fig. 3.12(a) for the laminate of aspect ratio = 1.0 and Fig. 3.12(b) for the laminate of aspect ratio = 1.5. In the case of aspect ratio = 1.0, the buckling load reaches the maximum at \( \theta = 45° \) as shown in Fig. 3.12(a). The reliabilities under three cases of load variations are also achieved the maximum at \( \theta = 45° \). As the load variation is larger, the change of reliability around 45° becomes very small. On the other hand, in the case of aspect ratio = 1.5, the reliability maximized design is different from the buckling load maximized design as already shown in Fig. 3.10. As the load variation increases, the reliability maximized design is shifted to larger angle. In case of COV = 0.05, the maximum is achieved at about 48°, while the buckling load reaches the maximum at about 42°.

As already shown in Section 3.6.4, a load variation has a large effect on the reliability.
Figure 3.12(b): Reliability for three cases of load variations, aspect ratio = 1.5.

Figure 3.13: Deterministic lower eigenvalues subject to uniaxial compression load, aspect ratio = 1.0.

The load components of the coordinate in the U-space have a large value at the design point. It means that the load ratio $N_y/N_z$ at the design point is different from the mean load condition. As the load variation is larger, the difference may be larger.

The state of the buckling mode at the maximum point has also an influence on the change of the reliability with respect to the load variation. The changes of the buckling load with respect to the load ratio are investigated in the range of $-0.1 \leq N_y/N_z \leq 0.2$, where $N_x^*$ is kept constant as 300 (GPa). In case of aspect ratio $= 1.0$, the buckling load factors of three laminates; $\theta = 40^\circ$, $45^\circ$, and $50^\circ$ are shown in Fig. 3.14(a). The $40^\circ$ and $50^\circ$ laminates have the same buckling load in this region. The $45^\circ$ laminate has the maximum buckling load regardless of the load ratio. The buckling load under the average load condition reaches the maximum at $45^\circ$ with a single eigenvalue as shown in Fig. 3.13. The buckling mode is not changed by the variation of the load ratio. Therefore, the $45^\circ$ laminate always has the maximum reliability regardless of the variation of the load ratio.
Fig. 3.14(b) shows the buckling load factors under the load ratio variation of four laminates; \( \theta = 40^\circ, 42.2^\circ, 45^\circ \) and \( 48^\circ \), where the \( 42.2^\circ \) laminate has the maximum buckling load under the mean load condition with double eigenvalues as shown in Fig. 3.10. The buckling mode is shifted at the point where the slope is changed. For the \( 42.2^\circ \) laminate, the mode is shifted at \( N_y/N_x = 0 \) and the buckling load is decreased rapidly when \( N_y \) is increased. On the other hand, the buckling load of the \( 45^\circ \) and the \( 48^\circ \) laminates are decreased smoothly. And then, these laminates have larger buckling load than the \( 42.2^\circ \) laminate. On the basis of \( N_y/N_x = 0.0 \), the changes of the buckling load of these laminates are smaller than that of the \( 42.2^\circ \) laminate. Therefore, these laminates have higher reliability than the \( 42.2^\circ \) laminate when the load variation is larger.
3.7 Summary

Reliability analysis of a simply-supported composite laminated plate subject to buckling is formulated under the condition that material constants and an orientation angle of each ply have some probabilistic variations as well as applied loads. In buckling analysis, an unsymmetric configuration due to variations of the random variables is considered. Then, the buckling failure is modeled as a series system consisting of eigen modes.

The mode reliability is evaluated through the FORM. During the mode reliability analysis, the changes of the random variables may cause mode crossings. In order to keep track of the intended mode, a mode tracking strategy which is based on eigenvector orthonormality is utilized. Then, the system reliability is evaluated by Ditlevsen's bounds.

Through numerical calculations, following conclusions are remarked.

1. The series system modeling is important for the reliability evaluation subject to buckling. This is because the most dominant failure mode does not always coincide with the mean buckling mode.

2. In comparison with Monte Carlo simulation, the suggested method is shown to have a sufficient accuracy for comparing reliabilities between different design candidates. Hence, the method can be utilized in reliability-based design.

3. Mode tracking strategy is shown to be effective to keep track of the intended mode during the mode reliability analysis.

4. Even if the mean stacking sequence is symmetric, an unsymmetric configuration should be considered for the reliability analysis.

5. It is shown that Young's modulus in the fiber direction and the ply orientation angle of each ply as well as applied load have dominant effect on the reliability.

6. Effects of buckling mode shifts on the reliability are investigated. Even if the buckling mode is repeated, the mode reliabilities of the corresponding failure mode is different from each other. Since the reliability of the series system is dominated by the lowest mode reliability, the design has lower reliability. On the other hand, if more than two mode reliabilities are well balanced, the design has higher reliability regardless of the mean buckling load.
Chapter 4

RELIABILITY-BASED OPTIMIZATION

4.1 Introduction

In Chapter 4, the reliability-based optimum design problem to maximize the buckling reliability in terms of the mean orientation angle of each ply is considered. The reliability is evaluated by the FORM which can be formulated as a nonlinear programming problem, which was described in Chapter 3. Therefore, the reliability-based design problem becomes a nested problem with two levels of optimization.

In order to determine the searching direction, the derivatives of the reliability is required. Therefore, the reliability sensitivity analysis with respect to the design variables is described. By utilizing the obtained results in the reliability analysis, the sensitivity of the reliability can be evaluated by small amount of additional calculations.

The obtained designs are compared with the deterministic buckling load maximization designs under several load cases. In comparing the states of the mode reliabilities between the deterministic design and the reliability-based design, the characteristics of the reliability-based design are investigated for two cases. One is the case where the deterministic design has a repeated buckling mode. The other is the case where the deterministic design has a single buckling mode.

The study on the reliability-based optimization subject to in-plane strength clarified that the reliability is increased as the number of the fiber axes increases. In the buckling reliability problem, the effect of the number of the fiber axes on the reliability is investigated by comparing the two types of angle-ply laminates. One is a bi-axial angle-ply $[+\theta/-\theta/ - \theta/ + \theta]$s with one fiber orientation variable and the other is a tetra-axial angle-ply $[+\theta_1/-\theta_1/-\theta_2/+\theta_2]$s with two orientation variables.
4.2 Formulation of Reliability-based Optimization

The problem is to find the laminate configuration such that the system reliability index may be maximized in terms of the mean ply orientation angle of each ply. The total number of the plies is set to be constant. The system reliability of the series system is approximated by Ditlevsen's upper bound, as described in Chapter 3. Therefore, $\beta_U$ is set as the objective function. The optimization problem is formulated as follows:

$$\text{Maximize : } \beta_U(u, t)$$
subject to : 
$$g_i(u = 0, t) \geq 0 \quad (i = 1, \ldots, N_f)$$

where $t$ denotes the design vector of ply orientation angles, and $u$ denotes the random vector in the reliability analysis. The inequality constraints mean that the origin of the $U$-space in the reliability analysis of the current design $t$ should lie in the safety region.

During the searching process, the constraints could sometimes be violated. It is not necessary to evaluate the system reliability of such a design exactly. Because a design...
lying in the failure region is never selected as an optimum design. Therefore, the objective function is replaced by the mode reliability corresponding to the mean buckling mode which has the negative value. The replacement may cause discontinuity of the objective function at the point where the sign of the constraints is changed to the positive. However, this discontinuity could not bring about a serious problem. Because the optimization algorithm immediately kicks back the design variable to the safety region. If the replacement still causes the problem, the design requirements will not be suitable, and the reliability-based design may have very low reliability. In such a case, it is better to modify the design requirement.

The reliability-based optimization problem becomes a nested problem with two levels of optimization, i.e., the main design problem and the reliability analysis. The calculation flow is shown in Fig. 4.1.

4.3 Sensitivity Analysis

The gradient of the reliability index is required to obtain the searching direction of the optimization algorithm. The derivative of the system reliability (Ditlevsen's upper bound; \( \beta_U \) in Eq. (3.11)) with respect to the design variable \( t_k \) is written as follows, Sørensen (1986) and Enevoldsen (1991):

\[
\frac{\partial \beta_U}{\partial t_k} = \frac{\partial \beta_U}{\partial P_U} \left( \sum_{i=1}^{2N_i} \frac{\partial P_i}{\partial \beta_i} \cdot \frac{\partial \beta_i}{\partial t_k} - \sum_{i=2}^{2N_i} \frac{\partial P_{ij}}{\partial t_k} \right) \tag{4.2}
\]

where \( P_{ij} \) is the selected component at the last part of Eq. (3.11).

\[
P_{ij} = \max_{j<i} P_{ij}
\]

The derivative of the mode reliability index \( \beta_i \) with respect to the mode failure probability \( P_i \) is written as follows:

\[
\frac{\partial \beta_i}{\partial P_i} = -\frac{1}{\phi(-\beta_i)} = -\sqrt{2\pi} \exp \left( \frac{\beta_i^2}{2} \right) \tag{4.3}
\]

Then, the derivative of the mode reliability index \( \beta_i \) with respect to the design variable \( t_k \) is:

\[
\frac{\partial \beta_i}{\partial t_k} = \frac{1}{|\nabla u_{g_i}(u^*, t)|} \frac{\partial g_i(u^*, t)}{\partial t_k} \tag{4.4}
\]

where \( \nabla u_{g_i}(u^*, t) \) is the gradient of the limit state function \( (g_i) \) at the design point \( (u^*) \), which is already obtained from the reliability analysis. \( \frac{\partial g_i}{\partial t_k} \) is easily obtained from the relation between the design variable \( t_k \) and the random variable \( u \).

\[
\frac{\partial g_i}{\partial t_k} = -\nabla g_i^T \frac{\partial u}{\partial t_k}
\]
Table 4.1: Variation of the material constants and the ply angle.

<table>
<thead>
<tr>
<th></th>
<th>$E_1$ (GPa)</th>
<th>$E_2$ (GPa)</th>
<th>$\nu_1$</th>
<th>$E_6$ (GPa)</th>
<th>$\theta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>181.0</td>
<td>10.3</td>
<td>0.28</td>
<td>7.17</td>
<td>---</td>
</tr>
<tr>
<td>c.o.v.*</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>s.d.† 5°</td>
</tr>
</tbody>
</table>

*: coefficient of variation, †: standard deviation.

The second term of Eq. (4.2), $\partial P_{ij}/\partial t_k$ is calculated as follows:

$$\frac{\partial P_{ij}}{\partial t_k} = \frac{\partial \Phi_2}{\partial \beta_i} \frac{\partial \beta_i}{\partial t_k} + \frac{\partial \Phi_2}{\partial \beta_j} \frac{\partial \beta_j}{\partial t_k} + 2 \frac{\partial \Phi_2}{\partial \rho_{ij}} \frac{\partial \rho_{ij}}{\partial t_k}$$  \hspace{1cm} (4.5)

where

$$\frac{\partial \Phi_2}{\partial \beta_i} = \phi_1(\beta_i) \cdot \Phi_1 \left( \frac{\beta_j - \rho_{ij} \beta_i}{\sqrt{1 - \rho_{ij}^2}} \right)$$  \hspace{1cm} (4.6)

Since $\partial \beta_i/\partial t_k$ is already obtained from Eq. (4.4), the first and the second terms of Eq. (4.5) are determined.

The derivatives with respect to the correlation coefficients of the third term of Eq. (4.5) require the second order derivatives of the limit state functions which take much computational costs. However, Sørensen (1986) has shown that it is often possible that the sensitivity analysis is approximated by neglecting the derivatives of the correlation coefficients. Therefore, Eq. (4.5) is approximated by neglecting the third term in this study.

### 4.4 Numerical Calculations

Through numerical calculations, the importance of the reliability-based design is demonstrated by comparing with the deterministic buckling load maximization design. Then, the characteristics of the reliability-based design and the dominant factors on the reliability are investigated. Finally, the effect of the number of fiber axes on the reliability is investigated.

Numerical examples are given for a simply supported 8 ply angle-ply laminated plate $[+\theta_1/-\theta_1/-\theta_2/+\theta_2]_S$ of aspect ratio $R = 2.0$ which is made up with graphite/epoxy (T300/5208) under several load conditions of bi-axial compression. The mean orientation angles $\bar{\theta}_1$ and $\bar{\theta}_2$ are treated as design variables.

This stacking sequence is selected to reduce the effect of the bending-twisting coupling terms $D_{16}$ and $D_{26}$. When the terms are neglected, it is known that the buckling load maximization design yields the bi-axial angle-ply laminate $[\pm\theta]_S$ which consists of only one orientation angle variable, Miki and Sugiyama (1993).

The material properties and the orientation angles of individual layers are assumed to be normally distributed. Their mean values and the coefficients of variations are listed in Table 4.1.
4.4. NUMERICAL CALCULATIONS

The applied load is standardized here for the plate dimensions as follows, Grenestedt (1990) and Miki and Sugiyama (1993):

\[ N^* = \frac{12a^2}{\pi^2h^3R^2}N \]

where \(a\) and \(b\) are the plate length and the plate width, respectively, \(R\) is the plate aspect ratio, \(R = a/b\), and \(h\) is the plate thickness. To investigate the effect of the applied load conditions on the reliability, seven cases of load conditions are considered, where the mean load ratio \(N_y/N_x\) is varied from 0.0 to 2.0, as listed in Table 4.2. The mean applied load is set to 0.8 times of the maximum buckling load of the bi-axial angle-ply plate which is listed in Table 4.3 (a).

The standard deviations of the applied loads are set as follows:

\[
\begin{align*}
SD(N_x) &= 0.05 \times \bar{N}_x, & COV(N_x) &= 0.05 \\
SD(N_y) &= \begin{cases} 0.025 \times \bar{N}_y & (\text{if } \bar{N}_y \neq 0) \\ 0.025 \times 0.25 \times \bar{N}_x & (\text{if } \bar{N}_y = 0) \end{cases} \\
SD(N_{xy}) &= 0.5SD(N_x)
\end{align*}
\]

In the case of \(N_y/N_x = 0.00\), the ratio of standard deviations of \(SD(N_x)/SD(N_y)\) is set to the same value as that in the case of \(N_y/N_x = 0.25\).

In the buckling analysis, the number of the assumed series of the displacement in Eq. (2.19) is set to \(M = N = 10\). The matrix size of the two sets of eigenvalue equations, \(m + n = \text{even}\) and \(m + n = \text{odd}\), is \(50 \times 50\). Hence, there exist 100 eigen modes totally. However, the maximum number of failure modes is limited to \(N_f = 6\). At most three eigen modes from the minimum eigenvalue are selected as failure modes in the ascending order for each eigenvalue problem. Because the failure modes corresponding to the higher eigenvalues at the mean value point design have relatively higher reliability and hence the contribution to the system reliability is negligible. Therefore, the failure modes whose mean eigenvalues are higher than 2.5 times as large as the maximum buckling load factor at the mean value point design are ignored to evaluate the system reliability in this thesis.

4.4.1 Bi-axial and Tetra-axial Angle-ply

In order to investigate the effect of the laminate configuration on the reliability, two types of the laminates are considered. One is a bi-axial angle-ply laminate with one fiber orientation variable \([+\theta/ -\theta/ -\theta/ +\theta]_s\) shown in Fig. 4.2(a), and the other is a tetra-axial angle-ply laminate with two fiber orientation variables \([+\theta_1/ -\theta_1/ -\theta_2/ +\theta_2]_s\) as shown in Fig. 4.2(b).

As the bi-axial laminate has only one design variable, the reliability maximization or the deterministic buckling load maximization is performed by one dimensional search such as a golden section method. In this case, the sensitivity analysis is not required. On the other hand, the tetra-axial laminate has two design variables, and thus the optimizations are
Table 4.2: Mean of applied load.

<table>
<thead>
<tr>
<th>Load case</th>
<th>$\bar{N}_y/\bar{N}_x$</th>
<th>$\bar{N}_z$ (GPa)</th>
<th>$\bar{N}_y$ (GPa)</th>
<th>$\bar{N}_{xy}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>303.20</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>235.20</td>
<td>58.80</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>0.50</td>
<td>139.55</td>
<td>104.66</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>0.75</td>
<td>139.55</td>
<td>104.66</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td>114.67</td>
<td>114.67</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td>1.50</td>
<td>84.27</td>
<td>126.41</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>2.00</td>
<td>66.50</td>
<td>133.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Figure 4.2(a): Bi-axial angle-ply $[+\theta/-\theta/-\theta/+\theta]_s$.

Figure 4.2(b): Tetra-axial angle-ply $[+\theta_1/-\theta_1/-\theta_2/+\theta_2]_s$. 
4.4. NUMERICAL CALCULATIONS

performed by sequential quadratic programming method (SQP) which is known as one of the most powerful tool of a nonlinear programming problem. The detail of SQP algorithm is described in Appendix A.

4.4.2 Deterministic Optimization

First, the deterministic optimal designs maximizing the buckling load are obtained for both types of the laminates. The problems are solved under the applied compressive load of $N_x = 1$(GPa) with varying $N_y/N_x$. The problem is formulated as follows:

$$\text{Maximize : } \lambda_{\min}(\theta)$$

subject to : $0^\circ \leq \theta \leq 90^\circ$

The number of the layers are set to be constant (8 plies) for both types of the laminate. Design variables are ply orientation angles for both types. The bi-axial angle-ply has only one variable; $\theta$, while the tetra-axial angle-ply has two variables; $\theta_1$ and $\theta_2$. The problem has only the side constraints of the orientation angles.

The buckling loads and the second smallest eigenvalues of the deterministic optimal designs are listed in Table 4.3. The maximum buckling load of both types of the laminates are found to be almost the same. Concerning the optimal laminate configuration of the bi-axial angle-ply, the orientation angle becomes larger as the load ratio $N_y/N_x$ increases. On the tetra-axial angle-ply, the difference in the ply orientation angles between the surface layer and the mid-plane layer becomes large as the load ratio $N_y/N_x$ increases. And the orientation angle of the surface layer is larger than that of the mid-plane layer except for the case of $N_y/N_x = 0.00$.

For the bi-axial angle-ply laminated plate, the changes of the buckling load in terms of the orientation angle $\theta$ is shown in Fig. 4.3. The points where the slopes change suddenly correspond to the shifted points of buckling modes. Except for the case of the uniaxial

![Figure 4.3: Buckling load of bi-axial angle-ply laminated plate, $[+\theta/ -\theta/ -\theta/ +\theta]$s.](image)
Table 4.3: Deterministic buckling load maximization design.

(a) Bi-axial angle-ply.

<table>
<thead>
<tr>
<th>$N_y/N_x$</th>
<th>Ply angle: $\theta$ (deg)</th>
<th>Buckling load (GPa)</th>
<th>Second eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>45.0</td>
<td>379.0</td>
<td>417.4</td>
</tr>
<tr>
<td>0.25</td>
<td>52.8</td>
<td>294.0</td>
<td>294.0</td>
</tr>
<tr>
<td>0.50</td>
<td>61.8</td>
<td>221.1</td>
<td>221.4</td>
</tr>
<tr>
<td>0.75</td>
<td>66.9</td>
<td>174.4</td>
<td>174.5</td>
</tr>
<tr>
<td>1.00</td>
<td>70.4</td>
<td>143.3</td>
<td>143.5</td>
</tr>
<tr>
<td>1.50</td>
<td>75.2</td>
<td>105.3</td>
<td>105.3</td>
</tr>
<tr>
<td>2.00</td>
<td>78.5</td>
<td>83.1</td>
<td>83.2</td>
</tr>
</tbody>
</table>

(b) Tetra-axial angle-ply.

<table>
<thead>
<tr>
<th>$N_y/N_x$</th>
<th>Ply angles</th>
<th>Buckling load (GPa)</th>
<th>Second eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>$\theta_1$</td>
<td>44.8</td>
<td>379.1</td>
</tr>
<tr>
<td>0.25</td>
<td>52.8</td>
<td>293.9</td>
<td>294.1</td>
</tr>
<tr>
<td>0.50</td>
<td>62.1</td>
<td>221.3</td>
<td>221.4</td>
</tr>
<tr>
<td>0.75</td>
<td>67.5</td>
<td>174.5</td>
<td>174.6</td>
</tr>
<tr>
<td>1.00</td>
<td>71.3</td>
<td>143.5</td>
<td>143.5</td>
</tr>
<tr>
<td>1.50</td>
<td>76.5</td>
<td>105.5</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>80.0</td>
<td>83.2</td>
<td></td>
</tr>
</tbody>
</table>

compression ($N_y/N_x = 0$), the minimum and the second smallest eigenvalues are very close to each other. It shows that the deterministic buckling load maximization is achieved at the point where eigenvalues are repeated.

4.4.3 Reliability-based Design of Bi-axial Angle-ply

The reliability-based designs of bi-axial angle-ply are listed in Table 4.4. The system reliability increases as the load ratio $\tilde{N}_y/\tilde{N}_x$ increases. This is because the variation of $N_y$ becomes comparatively small as the load ratio $\tilde{N}_y/\tilde{N}_x$ increases.

Table 4.4: Reliability-based design of bi-axial angle-ply.

<table>
<thead>
<tr>
<th>$\tilde{N}_y/\tilde{N}_x$</th>
<th>Ply angle $\theta$ (deg)</th>
<th>Mode reliabilitys</th>
<th>System reliability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_1^1$ $\beta_1^2$ $\beta_2^1$ $\beta_2^2$</td>
<td>$\beta_1^{\text{odd}}$ $\beta_2^{\text{odd}}$ $\beta_1^{\text{odd}}$ $\beta_2^{\text{odd}}$</td>
<td>$\beta_L$ $\beta_U$</td>
</tr>
<tr>
<td>0.00</td>
<td>44.9</td>
<td>4.780</td>
<td>6.244</td>
</tr>
<tr>
<td>0.25</td>
<td>53.7</td>
<td>3.966</td>
<td>4.352</td>
</tr>
<tr>
<td>0.50</td>
<td>60.7</td>
<td>3.893</td>
<td>4.133</td>
</tr>
<tr>
<td>0.75</td>
<td>65.0</td>
<td>4.041</td>
<td>4.248</td>
</tr>
<tr>
<td>1.00</td>
<td>68.2</td>
<td>4.251</td>
<td>4.446</td>
</tr>
<tr>
<td>1.50</td>
<td>73.1</td>
<td>4.742</td>
<td>4.911</td>
</tr>
<tr>
<td>2.00</td>
<td>76.5</td>
<td>5.165</td>
<td>5.845</td>
</tr>
</tbody>
</table>
Table 4.5: Reliability of the deterministic optimal designs of bi-axial angle-ply.

<table>
<thead>
<tr>
<th>$N_y/N_x$</th>
<th>$\beta_{\text{even}}^1$</th>
<th>$\beta_{\text{even}}^2$</th>
<th>$\beta_{\text{even}}^3$</th>
<th>$\beta_{\text{odd}}^1$</th>
<th>$\beta_{\text{odd}}^2$</th>
<th>$\beta_{\text{odd}}^3$</th>
<th>System reliability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>3.759</td>
<td>4.688</td>
<td>6.879</td>
<td>4.032</td>
<td>5.670</td>
<td>8.208</td>
<td>3.510</td>
</tr>
<tr>
<td>0.50</td>
<td>4.095</td>
<td>3.697</td>
<td>5.497</td>
<td>4.158</td>
<td>4.391</td>
<td>6.903</td>
<td>3.438</td>
</tr>
<tr>
<td>0.75</td>
<td>4.378</td>
<td>3.485</td>
<td>5.365</td>
<td>4.232</td>
<td>4.391</td>
<td>7.123</td>
<td>3.273</td>
</tr>
<tr>
<td>1.50</td>
<td>5.137</td>
<td>4.089</td>
<td>7.987</td>
<td>4.739</td>
<td>5.242</td>
<td>12.138</td>
<td>3.914</td>
</tr>
<tr>
<td>2.00</td>
<td>5.547</td>
<td>5.184</td>
<td>11.728</td>
<td>4.693</td>
<td>7.461</td>
<td>16.961</td>
<td>4.531</td>
</tr>
</tbody>
</table>

Figure 4.4(a): Mode and system reliabilities of deterministic design of bi-axial angle-ply.

Figure 4.4(b): Mode and system reliabilities of reliability-based design of bi-axial angle-ply.
To investigate the importance of reliability, the reliability of the deterministic buckling load maximization design of the bi-axial angle-ply is evaluated, which is listed in Table 4.5. The difference between the reliability-based design and the deterministic optimal design is not so large, at most 2.2 degree in the case of $N_y/N_x = 1.00$. However, in this case, the system reliability is improved as much as 0.672.

In Tables 4.4 and 4.5, the most critical failure modes are denoted in bold type. The critical modes are different for the most cases, except for the cases of $N_y/N_x = 0.0$. The reliability difference between the critical mode and the next critical mode of the reliability-based design is much smaller than that of the deterministic design, except for the case of $N_y/N_x = 0.00$. For example, in the case of $N_y/N_x = 2.00$, the difference of the reliability-based design is 0.09 between $\beta_{\text{even}}^1$ and $\beta_{\text{odd}}^1$. On the other hand, the difference of the deterministic design is 0.491 between $\beta_{\text{even}}^2$ and $\beta_{\text{odd}}^1$.

Except for the load cases of $N_y/N_x = 0.00$, the buckling load is very close to the second smallest eigenvalue in the deterministic design, as shown in Table 4.3. The dominant mode and the system reliabilities of both the deterministic and the reliability-based designs are shown in Figs. 4.4(a) and 4.4(b), respectively. In the deterministic design, the system reliability is dominated by a single failure mode. However, the reliability-based design have two dominant modes and the mode reliabilities are close to each other. Therefore, the mode reliabilities are well balanced and hence have higher system reliability.

For the load case of $N_y/N_x = 0.0$, the system reliability of the deterministic design is almost the same as that of the reliability-based design. Since buckling mode of the deterministic design is much separated from the second smallest eigenvalue as shown in Table 4.3, the buckling mode will be a single critical failure mode. Concerning the limit state function corresponding to the failure mode, the deterministic optimal design gives the largest value of all the design candidates. That's why the deterministic design has a relatively higher reliability.

### 4.4.4 Reliability-based Design of Tetra-axial Angle-ply

The reliability-based designs are listed in Table 4.6. The system reliability attains higher value as the load ratio increases, as in the case of the bi-axial angle-ply. Concerning the laminate configuration, the angle difference between the surface and the mid-plane layers is larger than the deterministic design. For example, in the case of $N_y/N_x = 1.0$, the angle difference of the deterministic design is 6.3°. On the other hand, the difference of the reliability-based design is 30.9°. Moreover, for the load cases of $0.25 \leq N_y/N_x \leq 0.75$, the orientation angle of the surface layer is smaller than those of the mid-plane layer. Such a property is completely opposite to the deterministic optimal design.

In the case of $N_y/N_x = 0.75$, two local optimal designs are found from several optimization runs, as shown in Table 4.6. One is $[+62.0°/ -62.0°/ -82.0°/ +82.0°]_s$; $\beta_U = 3.981$
Table 4.6: Reliability-based designs of tetra-axial angle-ply.

(a) System reliability.

<table>
<thead>
<tr>
<th>( \bar{N}_y / \bar{N}_x )</th>
<th>Ply angles ( \bar{\theta}_1 ) (deg)</th>
<th>System reliability</th>
<th>( \beta_L )</th>
<th>( \beta_U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>42.6</td>
<td>56.4</td>
<td>3.682</td>
<td>3.682</td>
</tr>
<tr>
<td>0.25</td>
<td>51.9</td>
<td>60.8</td>
<td>3.745</td>
<td>3.744</td>
</tr>
<tr>
<td>0.50</td>
<td>57.7</td>
<td>76.0</td>
<td>3.866</td>
<td>3.866</td>
</tr>
<tr>
<td>0.75</td>
<td>62.0</td>
<td>82.0</td>
<td>3.981</td>
<td>3.981</td>
</tr>
<tr>
<td>1.00</td>
<td>68.6</td>
<td>41.7</td>
<td>3.941</td>
<td>3.941</td>
</tr>
<tr>
<td>1.50</td>
<td>73.0</td>
<td>42.1</td>
<td>4.374</td>
<td>4.374</td>
</tr>
<tr>
<td>2.00</td>
<td>81.4</td>
<td>62.7</td>
<td>5.529</td>
<td>5.529</td>
</tr>
</tbody>
</table>

(b) Mode reliabilities.

<table>
<thead>
<tr>
<th>( \bar{N}_y / \bar{N}_x )</th>
<th>( \beta_{\text{even}}^1 )</th>
<th>( \beta_{\text{even}}^2 )</th>
<th>( \beta_{\text{even}}^3 )</th>
<th>( \beta_{\text{odd}}^1 )</th>
<th>( \beta_{\text{odd}}^2 )</th>
<th>( \beta_{\text{odd}}^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>4.782</td>
<td>6.020</td>
<td>7.992</td>
<td>3.857</td>
<td>6.376</td>
<td>9.397</td>
</tr>
<tr>
<td>0.25</td>
<td>4.045</td>
<td>4.582</td>
<td>6.685</td>
<td>4.016</td>
<td>5.529</td>
<td>7.946</td>
</tr>
<tr>
<td>0.50</td>
<td>4.139</td>
<td>4.357</td>
<td>6.152</td>
<td>4.463</td>
<td>5.090</td>
<td>7.468</td>
</tr>
<tr>
<td>0.75</td>
<td>4.235</td>
<td>4.419</td>
<td>6.258</td>
<td>4.809</td>
<td>5.092</td>
<td>7.862</td>
</tr>
<tr>
<td>1.00</td>
<td>4.209</td>
<td>4.431</td>
<td>7.637</td>
<td>4.885</td>
<td>5.721</td>
<td>10.047</td>
</tr>
<tr>
<td>1.50</td>
<td>4.629</td>
<td>4.856</td>
<td>8.888</td>
<td>4.820</td>
<td>6.471</td>
<td>11.923</td>
</tr>
<tr>
<td>2.00</td>
<td>5.513</td>
<td>5.935</td>
<td>11.363</td>
<td>5.609</td>
<td>8.087</td>
<td>15.465</td>
</tr>
</tbody>
</table>

Figure 4.5: Buckling contour plot in case of \( \bar{N}_y / \bar{N}_x = 0.75 \).
and the other is \([+68.6^\circ/-68.6^\circ/-41.7^\circ/+41.7^\circ]\); \(\beta_U = 3.941\). Though both designs have almost the same reliability, the laminate configurations are completely different. Note that other load cases may possibly have other local optima which the optimization process has not found yet.

It is investigated how the mean buckling load and the system reliability are varied around these optimal designs in terms of the ply orientation angles. Fig. 4.5 illustrates the contour plot of the mean buckling load factor, where the applied load is equal to 0.8 times of the maximum buckling load which is also the mean applied load for the reliability analysis. Therefore, the maximum buckling load factor is 1.25. This figure shows only the safety region where the buckling load factor is greater than or equal to 1.0. The black circle indicates the buckling load maximum design and the white circles correspond to the two local optima of the reliability-based design.

The system reliability is investigated in the rectangular region \((58^\circ \leq \theta_1 \leq 73^\circ, 30^\circ \leq \theta_2 \leq 90^\circ)\) indicated in Fig. 4.5. The contour plot of the system reliability is shown in Fig. 4.6. The black and white circles correspond to the deterministic and the reliability-based optimum designs, respectively.
4.4. NUMERICAL CALCULATIONS

Figure 4.7(b): Convergence history of the orientation angles, $\bar{N}_y / \bar{N}_x = 1.0$

Figure 4.8: Comparison of the system reliabilities of tetra-axial angle-ply.

Note that, in Fig. 4.5, the buckling load changes gradually around the reliability-based designs (the white circles). On the other hand, the change of the buckling load is sharper around the deterministic design (the black circle). It means that the deterministic optimal design is sensitive to the variation of the ply orientation angles. Therefore, the deterministic design has relatively lower reliability.

The convergence history of the reliability maximization for the load case of $\bar{N}_y / \bar{N}_x = 1.0$ is shown in Figs. 4.7(a) and 4.7(b), where the initial design is the buckling load maximization design; ( [+71.3° / −71.3° / −65.0° / +65.0°]s ). Fig. 4.7(a) shows the history of the system reliability (Ditleven's upper bound; $\beta(\cdot)$) and three significant mode reliabilities; $\beta^1_{\text{even}}$, $\beta^2_{\text{even}}$ and $\beta^1_{\text{odd}}$. Fig. 4.7(b) illustrates the history of the ply orientation angles; $\theta_1$ and $\theta_2$. In the initial stage, one critical mode dominates the system reliability. As the optimization process proceeds, the mode reliabilities become close to each other. At the ninth iteration, the mode reliabilities are the closest. Then, at the final design ([+73.0° / −73.0° / −42.1° / +42.1°]s), the mode reliabilities make little difference due to the effect of the joint probability $P_{ij}$. 
Table 4.7: Reliability of the deterministic optimal designs of tetra-axial angle-ply.

<table>
<thead>
<tr>
<th>( \bar{N}_y/\bar{N}_x )</th>
<th>( \beta_{\text{even}}^1 )</th>
<th>( \beta_{\text{even}}^2 )</th>
<th>( \beta_{\text{even}}^3 )</th>
<th>( \beta_{\text{odd}}^1 )</th>
<th>( \beta_{\text{odd}}^2 )</th>
<th>( \beta_{\text{odd}}^3 )</th>
<th>( \beta_L )</th>
<th>( \beta_U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>4.740</td>
<td>6.293</td>
<td>7.893</td>
<td>3.792</td>
<td>6.350</td>
<td>9.313</td>
<td>3.614</td>
<td>3.614</td>
</tr>
<tr>
<td>0.25</td>
<td>3.748</td>
<td>4.697</td>
<td>6.893</td>
<td>4.035</td>
<td>5.681</td>
<td>8.224</td>
<td>3.503</td>
<td>3.503</td>
</tr>
<tr>
<td>0.50</td>
<td>4.059</td>
<td>3.740</td>
<td>5.575</td>
<td>4.209</td>
<td>4.448</td>
<td>7.011</td>
<td>3.471</td>
<td>3.471</td>
</tr>
<tr>
<td>0.75</td>
<td>4.318</td>
<td>3.538</td>
<td>5.496</td>
<td>4.267</td>
<td>4.222</td>
<td>7.320</td>
<td>3.325</td>
<td>3.325</td>
</tr>
<tr>
<td>1.00</td>
<td>4.607</td>
<td>3.635</td>
<td>6.014</td>
<td>4.504</td>
<td>4.405</td>
<td>8.521</td>
<td>3.440</td>
<td>3.440</td>
</tr>
<tr>
<td>1.50</td>
<td>5.152</td>
<td>4.307</td>
<td>8.448</td>
<td>5.060</td>
<td>5.583</td>
<td>12.608</td>
<td>4.143</td>
<td>4.143</td>
</tr>
<tr>
<td>2.00</td>
<td>5.143</td>
<td>5.624</td>
<td>12.185</td>
<td>5.213</td>
<td>7.943</td>
<td>17.360</td>
<td>4.913</td>
<td>4.912</td>
</tr>
</tbody>
</table>

Figure 4.9(a): Mode and system reliabilities of deterministic design of tetra-axial angle-ply.

However, the two designs are almost identical. It indicates that the reliability-based design has more than two critical modes whose mode reliabilities are close to each other.

To investigate the reason why the reliability-based designs are so different from the deterministic designs, the system reliability of both designs are compared in Fig. 4.8. The difference in the system reliability between the deterministic and the reliability-based designs becomes larger as the load ratio \( \bar{N}_y/\bar{N}_x \) increases. The system and the mode reliabilities of the deterministic designs are listed in Table 4.7.

The most critical failure modes are denoted in bold type in both Tables 4.6 and 4.7. The dominant mode and the system reliabilities of both the deterministic and the reliability-based designs are shown in Figs. 4.9(a) and 4.9(b), respectively. As similar as the bi-axial angle-ply, the reliability difference between the critical mode and the next critical mode of the reliability-based design is much smaller than that of the deterministic design, except for the case of \( \bar{N}_y/\bar{N}_x = 0.00 \) and \( \bar{N}_y/\bar{N}_x = 0.00 \). For example, in the case of \( \bar{N}_y/\bar{N}_x = 1.50 \), the difference of the reliability-based design is 0.03 between \( \beta_{\text{even}}^1 \) and \( \beta_{\text{odd}}^1 \). On the other hand, the difference of the deterministic design is 0.75 between \( \beta_{\text{even}}^2 \) and \( \beta_{\text{odd}}^1 \).
Figure 4.9(b): Mode and system reliabilities of reliability-based design of tetra-axial angle-ply.

Figure 4.10: Comparison between bi-axial and tetra-axial angle-ply laminates.

As similar as the bi-axial angle-ply, the reliability-based design attains higher system reliability by making balance of two dominant failure modes. This is in a good contrast with the deterministic design which yields a single dominant buckling mode.

### 4.4.5 Comparison between Bi-axial and Tetra-axial Angle-ply

Finally, the reliability of both types of laminates are compared in Fig. 4.10.

The reliability of the tetra-axial angle-ply laminate is larger than that of the bi-axial angle-ply laminate although the maximum buckling load is almost the same. Besides, the difference in the reliabilities becomes larger as the load ratio $N_y/N_x$ is increased. By stacking the different orientation angles on the surface layer and the mid-plane, the tetra-axial angle-ply attains the higher reliability. This tendency is in agreement with the reliability-based design subject to in-plane strength where the reliability becomes larger as the number of the orientation axes increases, Muromtsu et al. (1994). Moreover, this is opposite to the
deterministic buckling load maximization design which yields angle-ply laminate with only one orientation angle variable, Miki and Sugiyama (1993).

Further, it is noted from comparison of Tables 4.3 and 4.6 that the optimal values of the ply angles for the tetra-axial conditions are very different from each other between the deterministic and the reliability-based designs.

4.5 Summary

The reliability-based design approach is applied to the composite laminated plate subject to buckling load. The reliability is evaluated by modeling the plate as the series system consisting of significant eigen modes, and considering the unsymmetry of laminates due to the random variation of the material constants and the orientation angles.

Comparing the reliability-based design with the deterministic one, it is shown that the reliability-based design is different from the deterministic optimum design for almost all the load cases. Especially, difference in the ply orientation angles between the surface and the mid-plane layers of the reliability-based design is larger than that of the deterministic one. By investigating the mode reliabilities, the following characteristics of the reliability-based design are clarified.

1. When there are more than two critical buckling modes, the deterministic and the reliability-based optimum designs are much different from each other. The deterministic one has close eigenvalues for each critical modes, while the reliability-based one has close mode reliabilities for the critical modes. The well balanced mode reliabilities in the latter lead to a higher system reliability.

2. On the other hand, when the deterministic optimum design has only one dominant buckling mode, the reliability of the deterministic design is almost the same as that of the reliability-based design since the buckling mode is also a single critical failure mode for the reliability analysis.

3. In comparison of the bi-axial and tetra-axial angle-ply laminates, it is shown that the reliability is improved by increasing the number of the fiber axes. This tendency is in agreement with the reliability-based design subject to in-plane strength.
Chapter 5

EFFECT OF CORRELATION

5.1 Introduction

Chapters 2 and 3 showed that Young's modulus in the fiber direction, the ply orientation angle and applied load are dominant factors on the reliability of a laminated plate subject to buckling. The discussion, however, was limited in the case where the random variables are assumed to be independent of each other.

Under the actual situations, some random variables will be statistically dependent. That is, there exist some correlations between variables. A lot of studies show that the correlations have large effects on the reliability. For example, consider a simply-supported beam subject to two concentrated loads with variations. The reliability of the beam is changed significantly by correlations between the two loads, Thoft-Christensen and Baker (1986).

In this chapter, the effects of correlations on the reliability and the reliability-based design of the laminated composite plate subject to buckling are investigated. First, the transformation of the correlated random variables into the independent standardized normally distributed variables is discussed.

Then, effects of correlations on the reliability are investigated for the following three types of variables by numerical calculations.

- Young's moduli in the fiber directions between layers.
- Applied loads.
- Ply orientation angles between layers.

These variables were shown to have large effects on the reliability and the reliability-based design in previous chapters. In the three cases, the reliability analysis and the reliability-based design are performed under the condition that other variables are assumed to be deterministic.
5.2 Transformation into Standard Normal Distribution Space

In the FORM, random variables $X = (X_1, \cdots, X_n)^T$ must be transformed into an independent standard normal vector $U = (U_1, \cdots, U_n)^T$; $U = T^{-1}(X)$, as shown in Chapter 2. When the variables $X$ have non-normal distribution, Rosenblatt transformation, Rosenblatt (1952) has been suggested by Hohenbichler and Rackwitz (1981) as a good choice.

Here, discussion is limited to the normal distribution, since all the variables are assumed to be normally distributed in this thesis. Then, more simple transformation can be utilized.

When the variables $X$ are independent of each other, $U$ is obtained by the following standardization:

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, \quad i = 1, \cdots, n$$  \hspace{1cm} (5.1)

where $\mu_{X_i}$ and $\sigma_{X_i}$ are the mean and standard deviation of $X_i$, respectively.

Then, consider the case where the random variables have dependent normal distribution. The covariance between two variables $X_i$ and $X_j$ is defined as follows:

$$\text{Cov}[X_i, X_j] = E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})] = E[X_i \cdot X_j] - E[X_i] \cdot E[X_j]$$  \hspace{1cm} (5.2)

Then, the correlation coefficient $\rho_{X_i, X_j}$ between two variables, $X_i$ and $X_j$ is defined as follows:

$$\rho_{X_i, X_j} = \frac{\text{Cov}[X_i, X_j]}{\sigma_{X_i} \cdot \sigma_{X_j}}$$  \hspace{1cm} (5.3)

The correlation coefficient should satisfy the following relation:

$$-1 \leq \rho_{X_i, X_j} \leq 1$$  \hspace{1cm} (5.4)

The correlations between any two variables can be written by using covariance matrix $C_X$ defined as follows:

$$C_X = \begin{bmatrix}
\sigma_{X_1}^2 & \rho_{X_1, X_2} \sigma_{X_1} \sigma_{X_2} & \cdots & \rho_{X_1, X_n} \sigma_{X_1} \sigma_{X_n} \\
\rho_{X_2, X_1} \sigma_{X_2} & \sigma_{X_2}^2 & \cdots & \rho_{X_2, X_n} \sigma_{X_2} \sigma_{X_n} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{X_n, X_1} \sigma_{X_n} & \rho_{X_n, X_2} \sigma_{X_n} & \cdots & \sigma_{X_n}^2
\end{bmatrix}$$  \hspace{1cm} (5.5)

When the $C_X$ is a diagonal matrix, there is no correlations between two variables. Using the orthogonal transformation, $X$ is transformed into variables $Y$ without any correlation.

$$Y = A^T X$$  \hspace{1cm} (5.6)

where $A$ is transformation matrix whose row consists of an orthonormal eigenvector of $C_X$.

The covariance matrix of $Y$ becomes a diagonal matrix:

$$C_Y = A^T C_X A = \begin{bmatrix}
\sigma_{Y_1}^2 & 0 & \cdots & 0 \\
0 & \sigma_{Y_2}^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \sigma_{Y_n}^2
\end{bmatrix}$$  \hspace{1cm} (5.7)
The variance of the new variables $\sigma_Y^2$ is equal to an eigenvalue of $C_X$ and the mean of the variables $\mu_Y$ is calculated from Eq. (5.6) by substituting $\mu_X$ into $X$.

Then, by using Eq. (5.1), the variables $Y$ is transformed into the standard normal distribution. The transformation can be written in the following matrix form:

$$U = C_Y^{-1/2}(Y - \mu_Y) = (A^T C_X A)^{-1/2} A^T (X - \mu_X)$$

(5.8)

where $C_Y^{-1/2}$ is a diagonal matrix whose diagonal elements consist of $1/\sigma_Y$.

Note that there exist lower limits of the correlation coefficients $\rho_{X_i,X_j}$. For physical reason, the standard deviation of the new variable, $\sigma_Y$ must be a real number. That is, the original covariance matrix $C_X$ should be positive definite. When the correlation coefficients between more than three variables take the same value, the lower limit of negative value is set from the positive definite condition of the covariance matrix, not from Eq. (5.4).

For example, if all correlation coefficients take the same value between all random variables, the limitation can be expressed as follows:

$$-\frac{1}{n-1} \leq \rho \leq 1$$

(5.9)

where $n$ is the number of the random variables.

### 5.3 Numerical Calculations

Numerical calculations are given for the tetra-axial angle-ply laminated plate $[+\theta_1/-\theta_1/-\theta_2/+\theta_2]_s$ made up with graphite/epoxy (T300/5208) which was used in the previous chapter. The plate of aspect ratio $R = 2.0$ subject to biaxial compression is considered.

Analyses are performed under the same conditions as in Chapter 4. That is, the order of the series of the displacement function in Eq. (2.19) is set to $M = N = 10$ in the buckling analysis. In the reliability analysis, eigen modes whose eigenvalues at the mean value point design are less than 2.5 are selected as failure modes. The maximum number of the failure modes is set to 6. Even if the buckling mode of the mean value point design is much separated from other modes, the reliability can be evaluated through both the first modes of $m + n = \text{even}$ and $m + n = \text{odd}$.

In the reliability-based optimization, the mean orientation angles $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are treated as design variables.

In order to investigate the effect of the correlations of each variable on the reliability, the remaining material constants, orientation angles, or applied loads are treated as deterministic and constant values. The material constants are given in Table ?? in Chapter 2. Concerning the applied loads, mean values of Table 4.2 in Chapter 4 are treated as deterministic values, except for the case where the applied loads are considered as random variables.

In all the cases of correlated variables, reliability analyses are performed for the deterministic buckling load maximization design, as shown in Table 4.3 (b) in Chapter 4.
5.3.1 Correlation between Young’s Moduli in Fiber Directions

First, effect of correlation between Young’s moduli in the fiber directions on the reliability is investigated. Only $E_1$s of all layers are dependent random variables. Other material constants, angle plies and applied loads are set to be deterministic and constant values. Therefore, the number of random variables is eight.

The variance of $E_1$s of all plies are assumed to take the same value and their covariances are:

$$\text{Cov}(E_1) = 0.05 \tag{5.10}$$

The correlation coefficients between layers are also assumed to take the same values. In this case, the correlation coefficient $\rho_{E_1}$ is bounded from the positive definite condition of the covariance matrix, Eq. (5.9).

$$-1/7 \leq \rho_{E_1} \leq 1 \tag{5.11}$$

Reliability Analysis

The reliability analysis is performed for the deterministic buckling load maximization design $[+52.8\degree/-52.8\degree/-52.6\degree/+52.6\degree]$s under the load condition of $N_y/N_x = 0.25$. The changes of mode reliabilities and the system reliability in terms of the correlation coefficient $\rho_{E_1}$ are shown in Fig. 5.1. The deterministic design is found to have two dominant failure modes which are the first mode of both $m+n = \text{even}$ and $m+n = \text{odd}$. When the correlation $\rho_{E_1}$ is large, the variations of $E_1$ of all plies tend to take the same value and the reliability is decreased.

Though other load cases are not shown here, they have similar effects. That is, the deterministic designs have two dominant failure modes whose reliabilities are almost the same. As the correlation coefficients of $E_1$ between plies are increased, the system reliability is decreased.
5.3. NUMERICAL CALCULATIONS

Table 5.1: Reliability-based design of load case $N_y/N_x = 0.25$ where $E_1$s of all layers are correlated variables.

(a) Optimum orientation angle and system reliability.

<table>
<thead>
<tr>
<th>$\rho_{E_1}$</th>
<th>Ply angles</th>
<th>System reliability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta_1$ (deg)</td>
<td>$\theta_2$ (deg)</td>
</tr>
<tr>
<td>-0.1</td>
<td>52.99</td>
<td>52.66</td>
</tr>
<tr>
<td>0.5</td>
<td>52.74</td>
<td>52.80</td>
</tr>
<tr>
<td>0.9</td>
<td>52.67</td>
<td>52.99</td>
</tr>
</tbody>
</table>

*: Deterministic design : $(\theta_1, \theta_2) = (52.8^\circ, 52.6^\circ)$.

(b) Mode reliabilities of reliability-based design.

<table>
<thead>
<tr>
<th>$\rho_{E_1}$</th>
<th>$\beta_{\text{even}}^{1}$</th>
<th>$\beta_{\text{even}}^{2}$</th>
<th>$\beta_{\text{odd}}^{1}$</th>
<th>$\beta_{\text{odd}}^{2}$</th>
<th>$\beta_{\text{odd}}^{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>12.095</td>
<td>17.110</td>
<td>26.20</td>
<td>12.094</td>
<td>22.203</td>
</tr>
<tr>
<td>0.5</td>
<td>5.572</td>
<td>8.239</td>
<td>15.478</td>
<td>5.589</td>
<td>11.824</td>
</tr>
<tr>
<td>0.9</td>
<td>4.492</td>
<td>6.658</td>
<td>12.509</td>
<td>4.490</td>
<td>9.553</td>
</tr>
</tbody>
</table>

Buckling load can be described as a nonlinear function in terms of the Young’s modulus in the fiber direction, as shown in Chapter 2. However, the nonlinearity is very weak. It can be almost regarded as the linear function. Additionally, the buckling load is monotonously increased as $E_1$ increases.

The basic applications of the structural reliability theory, Thoft-Christensen and Baker (1986), already made clear that the reliability of the series system which has the linear relationship between system failure and the element strength is decreased as the correlations between the element strength increase. The correlation between Young’s moduli in the fiber direction in layer materials has the similar effects on the reliability.

Reliability-based Optimization

In case of $N_y/N_x = 0.25$, reliability is maximized in terms of two orientation angle variables under three cases of the correlation coefficients $\rho_{E_1}$. The obtained results are listed in Table 5.1. The difference between the reliability-based design and the deterministic design is negligible for all three cases of the correlation. Because the deterministic design has two dominant modes whose mode reliabilities take almost the same value as shown in Fig. 5.1.

Though other load cases are not shown here, the reliability-based designs are almost the same as the deterministic designs due to the same reason.

5.3.2 Correlation between Applied Loads

Next, consider the case where applied loads $(N_x, N_y, N_{xy})$ are the correlated variables and material constants and angle plies are set to be deterministic and constant values.

In Chapter 3, the variation of the shear force $N_{xy}$ is shown to have small effects on reliability. This is opposite to the reliability subject to in-plane strength where variations
of shear force is known to have a large effects on reliability, Murotsu et al. (1994). In order to confirm effects of variations of the shear force in addition to effects of the correlations, the following three cases are considered.

**CASE 1** $N_x$ and $N_y$ are treated as correlated random variables ($-1 \leq \rho_{N_x,N_y} \leq 1$), but $N_{zxy}$ is set to be constant:

$$\text{Cov}(N_x) = \text{Cov}(N_y) = 0.05 \quad (5.12)$$

**CASE 2** All of $N_x$, $N_y$ and $N_{zxy}$ are treated as random variables. $N_x$ and $N_y$ have correlation ($-1 \leq \rho_{N_x,N_y} \leq 1$), but $N_{zxy}$ is independent:

$$\text{Cov}(N_x) = \text{Cov}(N_y) = 0.05$$
$$\text{SD}(N_{zxy}) = \text{SD}(N_x)$$
$$\rho_{N_x,N_{zxy}} = \rho_{N_y,N_{zxy}} = 0.0 \quad (5.13)$$

**CASE 3** All of $N_x$, $N_y$ and $N_{zxy}$ are treated as correlated random variables, and the correlations between any two elements are equal ($-0.5 \leq \rho \leq 1$):

$$\text{Cov}(N_x) = \text{Cov}(N_y) = 0.05$$
$$\text{SD}(N_{zxy}) = \text{SD}(N_x)$$
$$\rho_{N_x,N_{zxy}} = \rho_{N_y,N_{zxy}}$$ \quad (5.14)

In cases 2 and 3, the standard deviation of the shear force is given as the same value as that of the axial force in $X$-direction, $N_x$.

**Reliability Analysis**

Reliability analyses are performed on four deterministic designs under load cases of $\bar{N}_y/\bar{N}_x = 0.25$, 0.5, 1.0 and 1.5. The changes of the system reliability in terms of the correlation coefficients are shown in Figs. 5.2(a) through 5.2(d). Here, the legends “CASE 1”, “CASE 2” and “CASE 3” correspond to the above definition. Similarly as the case of correlation of $E_1$, the reliability is decreased as the correlation increases. This is because the buckling load is a linear function of the applied load.

The changes of the mode reliabilities in terms of the correlation coefficients in CASE 1 are shown in Fig. 5.3. Though other cases are not shown here, similar results are obtained for all three cases of random variables. The changes are summarized for each load case as follows:

- In the design under $\bar{N}_y/\bar{N}_x = 0.25$, the first mode of $m+n =$ odd is dominant. As the correlation coefficient increases, the mode reliability of the first mode of $m+n =$ even approaches that of the first mode of $m+n =$ odd. Then, the two mode reliabilities take almost the identical value.
5.3 NUMERICAL CALCULATIONS

Figure 5.2(a): System reliability of the deterministic design in terms of correlation coefficients between loads, $\hat{N}_y/N_x = 0.25$.

Figure 5.2(b): System reliability of the deterministic design in terms of correlation coefficients between loads, $\hat{N}_y/N_x = 0.5$.

Figure 5.2(c): System reliability of the deterministic design in terms of correlation coefficients between loads, $\hat{N}_y/N_x = 1.0$. 

Figure 5.2(d): System reliability of the deterministic design in terms of correlation coefficients between loads, $\bar{N}_y/\bar{N}_x = 1.5$.

- In the design under $\bar{N}_y/\bar{N}_x = 0.5$, the second mode of $m+n = \text{even}$ is dominant when the correlation coefficient is less than $-0.7$. However, when the correlation coefficient is greater than $-0.6$, mode reliabilities of both the first modes of $m+n = \text{even}$ and $m+n = \text{odd}$ take almost the same value.

- In the designs under $\bar{N}_y/\bar{N}_x = 1.0$ and $\bar{N}_y/\bar{N}_x = 1.5$, the first mode of $m+n = \text{even}$ is dominant. As the correlation coefficient is increased, the mode reliability of the first mode of $m+n = \text{odd}$ approaches that of the first mode of $m+n = \text{even}$. Then, the two mode reliabilities take almost the identical value.

In all load cases, the reliability takes almost the same value for three cases of correlations. It means that the effect of shear load on reliability is small.

Reliability-based Optimization

The reliability-based design is obtained for the four load cases, where the two axial compression loads are correlated random variables but the shear load is deterministic.

The changes of the optimum ply orientation angles $\theta_1$ and $\theta_2$ in terms of the correlation coefficient $\rho_{N_x,N_y}$ is shown in Figs. 5.4(a) through 5.4(d). In all load cases, the optimum orientation angle approaches the deterministic design as the correlation coefficient is larger. In case of $\bar{N}_y/\bar{N}_x = 0.5$, the reliability-based design is identical to the deterministic design when the correlation coefficient is larger than $-0.6$. This is because the deterministic design has two dominant mode reliabilities whose mode reliabilities are identical in case of $\rho_{N_x,N_y} \geq -0.6$.

Variations of applied loads yield changes in the resultant load direction which have large effects on the reliability. However, when the correlation coefficient $\rho_{N_x,N_y}$ is large, variations
Figure 5.3: Mode reliabilities of the deterministic designs in terms of correlation coefficients between loads under CASE 1.

Figure 5.4(a): Optimal orientation angles of reliability-based design subject to correlated loads, $\bar{N}_y/\bar{N}_x = 0.25$. 
Figure 5.4(b): Optimal orientation angles of reliability-based design subject to correlated loads, $\bar{N}_y/\bar{N}_x = 0.5$.

Figure 5.4(c): Optimal orientation angles of reliability-based design subject to correlated loads, $\bar{N}_y/\bar{N}_x = 1.0$.

Figure 5.4(d): Optimal orientation angles of reliability-based design subject to correlated loads, $\bar{N}_y/\bar{N}_x = 1.5$. 
5.3. NUMERICAL CALCULATIONS

Figure 5.5(a): The change of reliability of the deterministic design in terms of correlation coefficient between ply orientation angles, $N_y/N_x = 0.25$.

of resultant load direction becomes small. Therefore, the reliability-based optimal design approaches the deterministic design.

5.3.3 Correlation between Ply Orientation Angles

Finally, effects of correlation between the ply orientation angles on the reliability and the reliability-based optimization are investigated. Only ply orientation angles of all layers are treated as dependent random variables. On the other hand, material constants and applied loads are set to be deterministic and constant values. Therefore, the number of random variables is eight.

The variance of orientation angles $\theta$ of all plies are assumed to take the same value and their standard deviations are set to

$$SD(\theta) = 5^\circ$$  \hspace{1cm} (5.15)

The correlation coefficients between layers are also assumed to take the same value. In this case, the correlation coefficient $\rho_\theta$ is bounded as follows:

$$-1/7 \leq \rho_\theta \leq 1$$  \hspace{1cm} (5.16)

Reliability Analysis

Reliability analyses are performed on the two deterministic designs under the load cases of $N_y/N_x = 0.25$ and $0.5$. The changes of the system reliability and the dominant mode reliabilities in terms of correlation coefficient are shown in Figs. 5.5(a) and 5.5(b) for the cases of $N_y/N_x = 0.25$ and $0.5$, respectively. The system reliabilities have a peak around the value $0.5 \sim 0.6$ of the correlation coefficient $\rho_\theta$. This tendency is completely different from the above two cases of Young’s moduli in the fiber directions and applied loads. This
Figure 5.5(b): The change of reliability of the deterministic design in terms of correlation coefficient between ply orientation angles, $N_y/N_x = 0.5$.

Figure 5.6: Reliability of two ply laminate $[43.6^\circ_2]$ in terms of correlation coefficient between ply orientation angles.

is attributed to the fact that the relation between buckling load and ply orientation angle is strongly nonlinear.

Reliability of two ply laminates, $[+\theta_2]$

The nonlinearity effects of the reliability is illustrated through the following simple laminate. Consider the simply supported composite plate which consists of two plies with one orientation angle variable $[+\theta_2]$ subject to biaxial compression load with $N_y/N_x = 0.25$. The top and the bottom ply orientation angles are treated as correlated random variables, with the same mean values. The standard deviation is set to $SD(\theta) = 5^\circ$. Material constants and applied loads are assumed to be constant.

The reliability analysis is performed under the correlation coefficient $-1 \leq \rho_\theta \leq 1$ for the deterministic buckling load maximized design whose orientation angle is $\theta = 43.6^\circ$. 
Here, the applied load is set to 0.8 times of the maximum buckling load. The change of the reliability in terms of correlation is shown in Fig. 5.6. The first mode of $m + n = \text{odd}$ and the first mode of $m + n = \text{even}$ are dominant modes when $\rho_\theta$ is less than $-0.5$. In other regions, the first mode of $m + n = \text{even}$ is dominant. In this simple laminate, the reliability has a peak at $\rho_\theta = 0.6$.

The orientation angle corresponding to the design point of the first mode of $m+n = \text{even}$ is shown in Fig. 5.7. When the correlation coefficient is negative, the ply orientation angles of the two plies are different in the design point. However, the difference becomes small as the correlation coefficient is increased. When the correlation coefficient is larger than 0.6, both plies have the same orientation angle in the design point.

The limit state curve corresponding to $\lambda \equiv \text{even} = 1.0$ in the $X$-space is shown in Fig. 5.8. The abscissa axis $\theta_1$ is an orientation angle of the top layer and the ordinate axis $\theta_2$ is an orientation angle of the bottom layer. The black dot corresponds to the mean laminate, [43.6°]. As the reversed laminate has the same property, the limit state curve is symmetric in terms of the line of $\theta_1 = \theta_2$ whose slope is 45°. It is found that the limit state curve is strongly nonlinear with respect to the ply orientation angle.

In this case, the probability contour line can be expressed as ellipse whose principal axis declines 45 degree, as shown in Appendix C. In Fig. 5.8, the probability contour lines corresponding to the reliability indices are also plotted for the cases of the correlation coefficient $\rho_\theta = -0.8, -0.2, 0.2, 0.4$ and 0.9. A point of contact between the curves and each ellipse corresponds to the design point for each case of the correlation coefficient, which is shown in Fig. 5.7. Referring to Appendix C, it is found that the nonlinearity of the limit state curves is attributed to the fact that the reliability has a peak in terms of correlation coefficient.

Figure 5.7: Orientation angles of two plies corresponding to β point of the first mode of $m + n = \text{even}$. 
Reliability-based Optimization

Reliability-based optimizations are performed for the same two load cases; $N_y/N_x = 0.25$ and 0.5. The changes of the system reliability and the dominant mode reliabilities in terms of the correlation coefficient are shown in Figs. 5.9(a) and 5.9(b) for the cases of $N_y/N_x = 0.25$ and 0.5, respectively. Each reliability-based design has two dominant failure modes as in other conditions.

Optimal orientation angles are shown in Figs. 5.10(a) and 5.10(b) for the cases of $N_y/N_x = 0.25$ and 0.50, respectively. In case of $N_y/N_x = 0.25$, the optimal orientation angles are almost constant regardless of the values of correlation coefficient. The ply angles do not approach the deterministic design ($\theta_1 = 52.8^\circ, \theta_2 = 52.6^\circ$), even if the correlation coefficient is increased. On the other hand, in case of $N_y/N_x = 0.50$, the optimal orientation angles changes drastically with respect to the correlation coefficient. Especially, in the region of $\rho_\theta \leq 0.4$, the inner ply’s angle $\theta_2$ is larger than the outer ply’s angle $\theta_1$.

These results show that the correlation between ply orientation angles plays an important role in the reliability-based design due to the strong nonlinearity of buckling load in terms of the orientation angles.
Figure 5.9(a): The change of reliability of the reliability-based design in terms of correlation coefficient between ply orientation angles, $N_y/N_x = 0.25$.

Figure 5.9(b): The change of reliability of the reliability-based design in terms of correlation coefficient between ply orientation angles, $N_y/N_x = 0.5$.

Figure 5.10(a): Optimum orientation angles of the reliability-based design in terms of correlation coefficient between ply orientation angles, $N_y/N_x = 0.25$. 
5.4 Summary

In this chapter, effects of correlations between random variables on the reliability and the reliability-based design for the laminated composite plate subject to buckling are investigated. In numerical calculations, three cases of the random variables are considered. The first is correlations between the Young's modulus in the fiber direction of each ply, the second between applied loads, and the third between the orientation angles.

In the first and the second cases, the reliability is decreased as the correlations are increased. Moreover, the reliability-based design approaches the deterministic design, as the correlation coefficient between applied loads is increased. This is attributed to the fact that the variation of the resultant load direction is decreased. On the other hand, the reliability-based design is different from the deterministic design when the correlation coefficient between the loads is small, that is, the variation of the resultant load direction is large.

Finally, it is shown that the effect of the correlation between ply orientation angles is different from the above two cases. Since the buckling load is strongly nonlinear in terms of the ply orientation angles, the reliability has the peak at some positive value of the correlation coefficient. Then, the reliability-based design does not reach the deterministic design when the value of correlation coefficient is increased. This results show that the correlation between ply orientation angles plays an important role in the reliability analysis and the reliability-based design of fiber reinforced laminates.
Chapter 6

CONCLUSION

6.1 Summary of Thesis

In this thesis, structural reliability theory and the reliability-based optimization are applied to a composite laminated plate subject to buckling.

In order to evaluate the reliability, a buckling failure model is proposed by using lamination theory and reduced bending stiffness method together with structural systems reliability theory. The buckling load is obtained as a minimum eigenvalue in a deterministic analysis. However, an actual buckling mode may be different from the buckling mode corresponding to the minimum eigenvalue in the mean when design parameters have variations. Therefore, the buckling failure is modeled as a series system consisting of eigen modes for the reliability analysis of the composite laminated plate.

Following the formulation of the reliability analysis, effect of the variations in design parameters such as material constants, ply orientation angles and applied loads are investigated on reliability and optimum design.

A general summary of the individual chapters is given as follows.

Chapter 2

This chapter is devoted to the buckling analysis and the sensitivity analysis.

Even if the mean laminate sequence is symmetric, an actual laminate configuration is unsymmetric due to variations of material constants or ply orientation angles of layers. Therefore, buckling analysis for an unsymmetric laminate is required. For the purpose, the reduced bending stiffness method is used as an approximated analysis, and the Galerkin method is applied. The buckling analysis is used for evaluating the limit state function in the reliability analysis.

Then, sensitivities of buckling load with respect to the material constants and the orientation angle of each ply as well as the applied loads are formulated. Numerical calculations show that Young's modulus in the fiber direction and the orientation angle of each ply and applied loads have large sensitivities. Variations of these design parameters contribute
to large variation of buckling load. Accordingly, these parameters are expected to have dominant effect on the reliability subject to buckling.

It is known that the effect of the outer ply on the buckling load is generally larger than that of the inner ply. However, numerical results show that the sensitivity with respect to parameters of the inner ply is sometimes larger than that of the outer ply due to the existence of the bending-torsion coupling terms. Therefore, the inner ply parameters will also be an important factor on the reliability analysis.

The sensitivity analysis is also utilized in reliability analysis to obtain the searching direction of the reliability.

Chapter 3

In Chapter 3, reliability of a simply supported composite laminated plate subject to buckling is formulated under the condition where material constants and an orientation angle of each ply have some probabilistic variations as well as applied loads.

In the deterministic buckling analysis, the buckling load factor is obtained as a minimum eigenvalue. However, it is illustrated by a simple example that the failure mode corresponding to the mean buckling mode is not always dominant in the reliability analysis. In order to consider several eigen modes in the reliability analysis, buckling failure is modeled as a series system consisting of eigen modes. The mode reliability is evaluated through the FORM. During the mode reliability analysis, the changes of the random variables may cause mode crossings. In order to keep track of the intended mode, a mode tracking strategy is utilized. Then, the system reliability is approximated by Ditlevsen’s bounds.

Through numerical calculations, the proposed method is shown to have a sufficient accuracy in comparison with Monte Carlo simulation. Then, it is shown that the Young’s modulus in the fiber direction and the orientation angle of each ply as well as applied loads have dominant effects on the reliability. Finally, effects of buckling mode shifts on the reliability are investigated. When the buckling modes are repeated, the mode reliabilities of the corresponding failure modes are different from each other. Since the reliability of a series system is dominated by the lowest mode reliability, the design with the repeated eigenvalues has lower reliability. On the other hand, if more than two mode reliabilities are well balanced, the design has higher reliability regardless of the mean buckling load.

The series system modeling together with the mode tracking strategy is shown to be effective to evaluate the reliability subject to buckling load. The modeling can be applied to other reliability problems whose limit state functions are formulated as an eigenvalue problem such as a vibration problem.
Chapter 4

The reliability-based optimum design problem to maximize the buckling reliability in terms of mean orientation angle of each ply is formulated as a nested problem with two levels of optimization.

Through numerical calculations, the difference between the reliability-based optimum designs and the deterministic buckling load maximization designs is clarified under several load conditions.

Difference in the ply orientation angle between the surface layer and the mid-plane layer of the reliability-based design is larger than that of the deterministic design. This is because the larger difference reduces variations of the bending stiffness due to variations of orientation angles. Then, it is shown that the reliability is improved as the number of the fiber axes is increased. This tendency is in good agreement with the reliability-based design subject to in-plane strength.

When the deterministic design has a repeated buckling mode, the reliability-based design under the same load condition is much different from the deterministic design. The reliability-based design has close mode reliabilities corresponding to the critical failure modes. The well balanced mode reliabilities lead to a higher system reliability.

On the other hand, when the deterministic design has a single buckling mode, the difference between the reliability-based design and the deterministic design is small. The buckling mode corresponds to a single dominant failure mode in the reliability-based design. Therefore, the deterministic design has a relatively higher reliability because it has the highest value of the limit state function of the single dominant mode.

These results can be expanded to general buckling design problems. In buckling design problems, the deterministic buckling load maximization designs often have repeated buckling modes. Such designs may not have higher reliability because the mode reliabilities of the corresponding failure modes are not balanced. For the structural safety, it is expected that the design with well balanced mode reliabilities is selected as the reliability-based optimum design.

Chapter 5

In this chapter, effects of correlations between random variables on the reliability and the reliability-based design for the laminated composite plate subject to buckling are clarified.

In numerical calculations, three cases of the random variables are considered. The first is correlation between the Young's modulus in the fiber direction of each ply, the second between applied loads, and the third between the ply orientation angles. Through all the three cases, the reliability is changed drastically with respect to the correlation coefficients. It means that the identification of the correlation coefficients is important in the reliability analysis and the reliability-based design.
In the first and the second cases, the reliability is decreased as the correlations are increased. Moreover, the reliability-based design approaches the deterministic design as the correlation between applied loads is increased. This is because the variation of the resultant load direction is decreased. On the other hand, the reliability-based design is different from the deterministic design when the correlation between the load is small, that is, the variation of the resultant load direction is large.

Finally, it is shown that the effect of the correlation between ply orientation angles is different from the above two cases. The reliability has the peak at some positive value of the correlation coefficient. Then, the reliability-based design does not reach the deterministic design when the value of correlation coefficient is increased. This is because the buckling load is strongly nonlinear in terms of the ply orientation angles. This result shows that the correlation between ply orientation angles plays an important role on the reliability analysis and the reliability-based design.

6.2 Further Topics

In this thesis, discussions are limited to the case of simply supported boundary condition because the boundary condition has a lower buckling load than the clamped conditions. Additionally, it is widely used as an approximated boundary condition of a practical applications such as one partition of the stiffened panel. However, the study must be expanded to other boundary conditions such as clamped edges or combinations of simply supported and clamped ones.

A composite laminated plate has other failure modes such as in-plane failure and delamination. In order to evaluate the reliability of a laminated plate, these failure modes should be considered simultaneously. Additionally, strategy for identifying a dominant mode is required in the reliability analysis.

Reliability-based optimization is formulated as two levels of optimization problem in this thesis. The major drawback of the formulation is that the optimization process requires much calculation time. In fact, it takes more than hundreds of buckling analyses. Consequently, it is necessary to develop an efficient optimization algorithm.

Further, in order to evaluate reliability of an actual composite structural system such as a stiffener panel, it is required to combine the reliability analysis and the reliability-based design with finite element method.
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Appendix A

Sequential Quadratic Programming Method

Sequential quadratic programming method (SQP) is known as one of the most powerful method to solve the constrained nonlinear programming problem. This method is used in the reliability analysis and the reliability-based optimization in this thesis. The basic algorithm is explained here. SQP is also called a quasi-Newton method with constraints. In a quasi-Newton method, an objective function is approximated quadratically and the Hessian matrix is approximated from the first derivatives.

Optimal Condition

The general nonlinear optimization problem is formulated as follows:
Find the set of design variables, \( x = (x_1, x_2, \cdots, x_n)^T \) that will

Minimize: \( f(x) \)

subject to:
\[
\begin{align*}
    c_i(x) &= 0, \quad (i = 1, \cdots, m_e) \\
    c_i(x) &\geq 0, \quad (i = m_e + 1, \cdots, m)
\end{align*}
\]

where \( f(x) \) is the objective function and \( c_i(x) \) is the constraint functions. The \( m_e \) equality constraints \( c_i(x), i = 1, \cdots, m_e \) which must be satisfied at the optimum design. The \( m - m_e \) inequality constraints \( c_i(x), i = m_e + 1, \cdots, m \) impose a region which is feasible for the optimum design.

The local optimal solution corresponding to Eq. (A.1) is denoted by \( x^* \). When the gradient vector \( \nabla c_i(x^*) \) of the active constraints at the optimal solution \( x^* \) is linearly independent, there exists the vector set \( (x, u) = (x^*, u^*) \) which satisfies the following Kuhn-Tucker conditions. The active constraint set means the set which satisfy \( c_i(x) = 0 \):

\[
\begin{align*}
    \nabla f(x) - \sum_{i=1}^{m} u_i \nabla c_i(x) &= 0 \\
    c_i(x) &= 0, \quad (i = 1, \cdots, m_e) \\
    c_i(x) &\geq 0, \quad u_i \geq 0, \quad u_i c_i(x) = 0, \quad (i = m_e + 1, \cdots, m)
\end{align*}
\]

where \( u = \{u_1, u_2, \cdots, u_m\} \) is Lagrange's multipliers vector.
Quasi-Newton Method for Constrained Problem

First, consider the problem which contains only the equality constraints:

Minimize: $f(x)$
subject to: $c_i(x) = 0$ \hspace{1cm} (i = 1, \cdots, m_e) \hspace{1cm} (A.5)

Lagrangian for Eq. (A.5) is defined as follows:

$$L(x, u) = f(x) - \sum_{i=1}^{m} u_i c_i(x)$$ \hspace{1cm} (A.6)

The gradients and the Hessian matrix for this function are given by

$$\nabla_x L(x, u) = \nabla f(x) - \sum_{i=1}^{m} u_i \nabla c_i(x)$$ \hspace{1cm} (A.7)

$$\nabla_x^2 L(x, u) = \nabla^2 f(x) - \sum_{i=1}^{m} u_i \nabla^2 c_i(x)$$ \hspace{1cm} (A.8)

Consider the quadratic programming problem which obtains the next iteration point $x^{(k+1)}$ for the given point $x^{(k)}$:

Minimize: $\nabla f(x^{(k)})^T d + \frac{1}{2} d^T B^{(k)} d$
subject to: $c(x^{(k)}) + \nabla c(x^{(k)})^T d = 0$ \hspace{1cm} (A.9)

where $d$ is the displacement vector from $x^{(k)}$. The objective function is the quadratic approximation of Eq. (A.5) at the current point $x = x^{(k)}$ without $f(x^{(k)})$, and the constraint functions are linearly approximated. The matrix $B^{(k)}$ is the approximation of the Hessian of the Lagrangian $L$.

Kuhn-Tucker condition for the quadratic programming problem (A.9) is given by:

$$\nabla f(x^{(k)}) + B^{(k)} d - \nabla c(x^{(k)}) u = 0$$ \hspace{1cm} (A.10)

$$c(x^{(k)}) + \nabla c(x^{(k)})^T d = 0$$ \hspace{1cm} (A.11)

Introducing $d = x^{(k+1)} - x^{(k)}$ and $u = u^{(k+1)}$, these conditions can be expressed as follows:

$$\begin{bmatrix} B^{(k)} & -\nabla c(x^{(k)}) \\ \nabla c(x^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} x^{(k+1)} - x^{(k)} \\ u^{(k+1)} - u^{(k)} \end{bmatrix} = - \begin{bmatrix} \nabla_x L(x^{(k)}, u^{(k)}) \\ c(x^{(k)}) \end{bmatrix}$$ \hspace{1cm} (A.12)

Since the problem does not have any inequality constraints, the Kuhn-Tucker condition for the problem (A.5) can be described as the nonlinear equations with $n + m$ variables.

$$\nabla_x L(x, u) = 0$$ \hspace{1cm} (A.13)

$$c(x) = 0$$ \hspace{1cm} (A.14)

Applying the Newton method directly, the iteration equation is described as follows:

$$\begin{bmatrix} \nabla_x^2 L(x^{(k)}, u^{(k)}) & -\nabla c(x^{(k)}) \\ \nabla c(x^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} x^{(k+1)} - x^{(k)} \\ u^{(k+1)} - u^{(k)} \end{bmatrix} = - \begin{bmatrix} \nabla_x L(x^{(k)}, u^{(k)}) \\ c(x^{(k)}) \end{bmatrix}$$ \hspace{1cm} (A.15)
Eq. (A.12) just replaces the Hessian of Eq. (A.15) by $B^{(k)}$. If $B^{(k)} = \nabla_x^2 L(x^{(k)}, u^{(k)})$, the iteration of the Newton method (A.15) is equivalent to the optimal solution of the problem (A.9) and the corresponding Lagrange's multipliers.

If the matrix $B^{(k)}$ is updated to be a good approximation of the Hessian $\nabla_x^2 L(x^{(k)}, u^{(k)})$, the iteration method using quadratic programming problem (A.9) will have a good performance such as Newton method.

For the quasi-Newton method for an unconstrained problem, BFGS formula is used to update the matrix $B^{(k)}$:

$$B^{(k+1)} = B^{(k)} + \frac{y^{(k)}(y^{(k)})^T}{(y^{(k)})^T s^{(k)}} - \frac{B^{(k)} s^{(k)}(s^{(k)})^T B^{(k)}}{(s^{(k)})^T B^{(k)} s^{(k)}}$$  \hspace{1cm} (A.16)

where $s^{(k)}$ and $y^{(k)}$ are defined as follows:

$$s^{(k)} = x^{(k+1)} - x^{(k)}$$  \hspace{1cm} (A.17)

$$y^{(k)} = \nabla_x L(x^{(k+1)}, u^{(k+1)}) - \nabla_x L(x^{(k)}, u^{(k+1)})$$  \hspace{1cm} (A.18)

$B^{(k+1)}$ should be positive definite to solve the quadratic problem. For the constrained problem, however, Eq. (A.16) does not guarantee that $B^{(k+1)}$ is positive definite. For this purpose, the following condition should be satisfied:

$$(y^{(k)})^T s^{(k)} > 0$$  \hspace{1cm} (A.19)

Therefore, the updating formula is modified. First, a new variable $\tilde{y}^{(k)}$ is introduced as follows:

$$\tilde{y}^{(k)} = \theta y^{(k)} + (1 - \theta) B^{(k)} s^{(k)}$$  \hspace{1cm} (A.20)

where $\theta$ is defined by

$$\theta = \begin{cases} 1 & \text{if } (s^{(k)})^T y^{(k)} \geq 0.2 (s^{(k)})^T B^{(k)} s^{(k)} \\ 0.8 (s^{(k)})^T B^{(k)} s^{(k)} & \text{otherwise} \end{cases}$$  \hspace{1cm} (A.21)

Then, BFGS formula (A.16) is rewritten by using $\tilde{y}^{(k)}$ instead of $y^{(k)}$. That is,

$$B^{(k+1)} = B^{(k)} + \frac{\tilde{y}^{(k)}(\tilde{y}^{(k)})^T}{(\tilde{y}^{(k)})^T s^{(k)}} - \frac{B^{(k)} s^{(k)}(s^{(k)})^T B^{(k)}}{(s^{(k)})^T B^{(k)} s^{(k)}}$$  \hspace{1cm} (A.22)

Expanding this idea to the problem with inequality constraints, the quadratic programming approximation of the problem (A.1) is described as follows:

Minimize:  \hspace{1cm} $\nabla f(x^{(k)})^T d + \frac{1}{2} d^T B^{(k)} d$

Subject to:  \hspace{1cm} $c_i(x^{(k)}) + \nabla c_i(x^{(k)})^T d = 0, \quad (i = 1, 2, \ldots, m_e)$  \hspace{1cm} (A.23)

$\quad c_i(x^{(k)}) + \nabla c_i(x^{(k)})^T d \geq 0, \quad (i = m_e + 1, m_e + 2, \ldots, m)$

Dual method is applied to solve the subproblem (A.23).
Penalty Function and Line Search

The SQP method only has a local convergency. To guarantee the global convergency, the penalty function of Eq. (A.1) is introduced as follows:

\[ F_r(x) = f(x) + r \left[ \sum_{i=1}^{m_e} |c_i(x)| + \sum_{i=m_e+1}^{m} \min\{0, c_i(x)\} \right] \]  

(A.24)

where \( r \) is a penalty parameter. The term inside [ ] of Eq. (A.24) is equal to 0 if the solution is feasible and becomes larger as the solution is apart from the feasible region.

Assume that \((x^*, u^*)\) satisfies Kuhn-Tucker condition of the problem (A.1). If the penalty parameter \( r \) satisfies the following condition,

\[ r > \max\{|u^*_i| \mid i = 1, 2, \ldots, m\} \]  

(A.25)

then \( x^* \) becomes a stationary point of the function \( F_r \) under suitable conditions. Such a penalty function is called exact penalty function.

Assume that a penalty parameter \( r \) is a large constant number and the matrix \( B^{(k)} \) in the objective function of Eq. (A.23) is positive definite. In this case, the following inequality is satisfied for an optimal solution \( d^{(k)} \):

\[ F'_r(x^{(k)}, u^{(k)}) < 0 \]  

(A.26)

This shows that the direction vector \( d^{(k)} \) is a descent direction of the function \( F_r \) at \( x^{(k)} \).

Therefore, by conducting the line search along \( d^{(k)} \) direction, the suitable step size \( t^{(k)} > 0 \) satisfying the following inequality will be found:

\[ F_r(x^{(k)} + t^{(k)}d^{(k)}) < F_r(x^{(k)}) \]  

(A.27)

The next iteration point is determined as \( x^{(k+1)} = x^{(k)} + t^{(k)}d^{(k)} \).

It is only required that the next iteration point will decrease the objective function. The accurate minimization along the decreasing vector \( d^{(k)} \) is not necessary because the most important thing is to find the suitable step size \( t^{(k)} \) by the minimum effort of calculations.

Algorithm of SQP

The algorithm is summarized as follows:

**Step 1** Set the initial point \( x^{(0)} \), the positive definite matrix \( B^{(0)} \), the penalty parameter \( r > 0 \) and \( k = 0 \).

**Step 2** Obtain \((d^{(k)}, u^{(k+1)})\) by solving the quadratic programming problem (A.23).

**Step 3** Update the penalty parameter \( r \) which satisfies the condition (A.25).

**Step 4** Obtain the step size \( t^{(k)} \) satisfying the condition (A.27) by the line search along the direction \( d^{(k)} \). Set \( x^{(k+1)} = x^{(k)} + t^{(k)}d^{(k)} \).

**Step 5** Update the coefficient matrix \( B^{(k)} \) by using the formula (A.22).

**Step 6** If the Kuhn-Tucker condition is satisfied and the relative change in the penalty function (A.24) is small enough, \( x^{(k+1)} \) is regarded as the optimal solution and the iteration is terminated. Otherwise, set \( k = k + 1 \) and go back to Step 2.
Appendix B

Derivatives of Eigenvectors

Here, the derivatives of eigenvectors are explained, which is used in the sensitivity analysis for the repeated eigenvalue and the mode tracking.

Consider a general eigenvalue problem:

\[ [K - \lambda_i M] \phi_i = 0 \quad (i = 1, \ldots, N) \]  

(B.1)

where eigenvectors \( \phi_i \) are assumed to be \( M \)-orthonormal. That is

\[ \phi_j^T M \phi_i = \delta_{ij} \]  

(B.2)

The derivative of the eigenvectors with respect to a design variable \( X_j \) is obtained by taking the derivative of Eq. (B.1).

\[ (K - \lambda_i M) \frac{\partial \phi_i}{\partial x_j} = \left( \frac{\partial \lambda_i}{\partial x_j} M + \lambda_i \frac{\partial M}{\partial x_j} - \frac{\partial K}{\partial x_j} \right) \phi_j \]  

(B.3)

However, since the coefficient matrix of the left hand side \( (K - \lambda_i M) \) is a singular, Eq. (B.3) cannot be solved directly.

Nelson’s method, Nelson (1976), obtains an exact solution to Eq. (B.3). This method expresses the eigenvector derivative in terms of a particular solution \( P \) and a complementary solution \( C \phi_i \), where \( C \) is an unknown coefficient:

\[ \frac{\partial \phi_i}{\partial x_j} = P + C \phi_i \]  

(B.4)

Substituting Eq. (B.4) into Eq. (B.3) yields

\[ (K - \lambda_i M) P + C(K - \lambda_i M) \phi_i = F_i \]  

(B.5)

where \( F_i \) is defined as the vector of the right hand side in Eq. (B.3). Since the second term is equal to zero, Eq. (B.5) becomes

\[ (K - \lambda_i M) P = F_i \]  

(B.6)

The particular solution is found by identifying the component of the eigenvector \( \phi_i \) with the largest absolute value and constraining the derivative of that element to zero. If the \( k \)-th component takes the largest absolute value, then the corresponding components of \( P \)
and \( F_i \) are set to zero. That is, \( P_k = F_{ik} = 0 \). Moreover, the corresponding coefficients of the matrix \( (K - \lambda_i M) \) is modified as follows:

\[
\begin{align*}
(K - \lambda_i M)_{kl} &= 0, & (K - \lambda_i M)_{lk} &= 0, & (l \neq k) \\
(K - \lambda_i M)_{kk} &= 1
\end{align*}
\]  

(B.7)

Therefore, Eq. (B.6) can be written as follows:

\[
\begin{bmatrix}
(K - \lambda_i M)_{11} & \cdots & 0 & \cdots &(K - \lambda_i M)_{1N} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0 \\
\vdots & \cdots & \ddots & \ddots & \vdots \\
(K - \lambda_i M)_{N1} & \cdots & 0 & \cdots &(K - \lambda_i M)_{NN}
\end{bmatrix}
\begin{bmatrix}
P_1 \\
\vdots \\
0 \\
\vdots \\
P_N
\end{bmatrix}
= 
\begin{bmatrix}
F_{i1} \\
\vdots \\
0 \\
\vdots \\
F_{iN}
\end{bmatrix}
\]

(B.8)

\( P \) can be obtained directly to solve Eq. (B.8) because rank of this equation is \( N \). It is required to solve Eq. (B.8) repeatedly as many times as the number of design variables. However, once LU decomposition is performed, other solutions can be obtained only by forward and backward substitutions. Because the coefficient matrix of the left hand side is common in all design variables.

Then, the unknown coefficient \( C \) is obtained. Taking the partial derivative of Eq. (B.2) with respect to a design variable \( x_j \) yields

\[
2\phi_i^T M \frac{\partial \phi_k}{\partial x_j} + \phi_i^T \frac{\partial M}{\partial x_j} \phi_k = 0
\]

(B.9)

Substitution of Eq. (B.4) into Eq. (B.9) gives

\[
2\phi_i^T M P + 2C \phi_i^T M \phi_i + \phi_i^T \frac{\partial M}{\partial x_j} \phi_i = 0
\]

(B.10)

From the \( M \)-orthonormality of eigenvectors, the coefficient \( C \) is obtained as follows:

\[
C = -\phi_i^T M P - \frac{1}{2} \phi_i^T \frac{\partial M}{\partial x_j} \phi_i
\]

(B.11)

Thus, the eigenvector derivatives are obtained.
Appendix C

Correlated Normal Distribution

Consider the two dimensional correlated normaly distributed random vector \( \mathbf{X} = (X_1, X_2)^T \) whose standard deviations are the same, \( \sigma_1 = \sigma_2 = \sigma \), and the correlation coefficient is described as \( \rho \). In this case, reliability index can be expressed as follows:

\[
\beta = \min_{\mathbf{u} \in \mathbb{R}^n} (\mathbf{u}^T \mathbf{u})^{1/2} = \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ (\mathbf{x} - E[\mathbf{x}])^T \mathbf{C}_x^{-1} (\mathbf{x} - E[\mathbf{x}]) \right\}^{1/2} \tag{C.1}
\]

where \( \mathbf{C}_x \) is a covariance matrix:

\[
\mathbf{C}_x = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad \mathbf{C}_x^{-1} = \frac{1}{\sigma^2(1 - \rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \tag{C.2}
\]

The coordinate of design point is denoted as \( \mathbf{x}^* \) and a set of new variables \( \mathbf{y} = \mathbf{x}^* - E[\mathbf{x}] \) is introduced. Then, Eq. (C.1) can be written as follows:

\[
y_1^2 - 2\rho y_1 y_2 + y_2^2 = \beta^2 \sigma^2 (1 - \rho^2) \tag{C.3}
\]

Eq. (C.3) means that the probability contour line of \( \Phi_1^{-1}(\beta) \) is expressed as an ellipse whose center is located in the mean value point \((E[X_1], E[X_2])\) and whose principal axis declines 45 degree in the \( \mathbf{X} \)-space. The major axis declines 45° if \( \rho \) is positive. On the other hand, the major axis declines -45° if \( \rho \) is negative. When \( \rho \) is equal to zero, the locus can be expressed as a circle.

The ellipse can be expressed as the following canonical form after \(-45^\circ\) rotation.

\[
\frac{t_1^2}{\beta \sigma \sqrt{1 + \rho^2}} + \frac{t_2^2}{\beta \sigma \sqrt{1 - \rho^2}} = 1 \tag{C.4}
\]

where

\[
t_1 = \sqrt{2}(y_1 + y_2)/2, \quad t_2 = \sqrt{2}(-y_1 + y_2)/2 \tag{C.5}
\]

Ellipses corresponding to \( \beta = \text{constant} \) for \(-1 \leq \rho \leq 1\) is shown in Fig. C.1. The ellipses occupy the square whose side is \(2\beta\) in the center of \(E[\mathbf{X}]\).

Therefore, if a limit state surface can be expressed as a line, the relation between the reliability index and the correlation coefficient depends on the slope of the limit state surface, as shown in Fig. C.2.
Figure C.1: The locus of Eq. (C.3) for $\beta = \text{constant}$ under several values of $-1 \leq \rho \leq 1$.

Figure C.2: Reliability index for the linear limit state surface.