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Sales Revenue and Expected Profit of Monopsonist, Vanishing of Discontinuous Segment, & Tax Evasion.*

Shigeru Watanabe

§ 1. Introduction

Maximizing profit or expected profit will be a plausible objective of the taxpayer. Maximizing sales revenue will also be a plausible objective of the taxpayer, provided that a profit level doesn't fall below a certain given level.1)


The first purpose of this paper is to consider both the sales revenue and the expected profit as the objective of the taxpayer. To analyze the effect of considering the sales revenue as a component of the taxpayer's objective, we will deal with a monopsonist in a labour market. Other types of taxpayes will be examined in forthcoming papers (See Watanabe (1992.a, 1992.b, ))

In the next section, a simple model including sales revenue will be analyzed. In section 3, the effect of raising the tax rate or the penalty rate

* I would like to thank professors G. Suhama, S. Ito and Y. Tomita for their helpful comments.
1) See Baumol (1958).
on the quantity of the employed labour, the rate of the underreporting and
total tax collections will be examined. In section 4, further effects of
considering the sales revenue as a component of the taxpayer’s objective
will be analyzed. In section 5, declared sales revenue instead of true sales
revenue will be examined.

In section 6, another purpose of this paper will be analyzed without
considering the sales revenue as a component of the taxpayer’s objective
to make the analysis simple. The another purpose of this paper is to
show that even if the demand curve is kinked, the discontinuous segment
(, which allows marginal cost to change without price changes, ) may
vanish when the tax evasion is taken into consideration.

In the last section, main results will be summarized.

§ 2. Simple Model

The objective\(^1\) which is composed of the sales revenue and the expected
profit will be denoted as

\[ V = \alpha ph(x) + \beta E, \quad (1) \]

where \( \alpha, \beta > 0, h(x) (h'(x) > 0, h''(x) < 0) \) is the output level which is
assumed to be increasing along with the quantity of labour \( x \) employed,
p is the price level and \( E \) is the expected profit when the underreporting of
gross revenues is taken into consideration.

When the tax evasion by underreporting gross revenues or sales reve-
 nues is not detected, the profit will be denoted as

\[ ph(x) - w(x) x - t \{ (1-\epsilon) ph(x) - w(x) x \}, \quad (2) \]

where \( w \) is the wage rate \( (w'(x) > 0, w''(x) > 0) \), which is assumed to be
an increasing function of the amount employed since taxpayer is assumed
to be a monopsonist, \( t \) is the tax rate and \( \epsilon \) is the rate of the underrepo-

\(^1\) The case in which the sales revenue is not considered as a component of
the objective has already been analyzed in Watanabe (1989,a)
rting.

On the other hand, when the tax evasion is detected, the profit will be denoted as
\[
ph(x) - w(x)x - t \{(1-\varepsilon)ph(x) - w(x)x\} - Ft\varepsilon ph(x),
\]
(3)
where F is the penalty rate.

From (2) and (3) we have the following expected profit;
\[
E = (1-q(\varepsilon)) \left[ ph(x) - w(x)x - t \{(1-\varepsilon)ph(x) - w(x)x\} \right] \\
+ q(\varepsilon) \left[ ph(x) - w(x)x - t \{(1-\varepsilon)ph(x) - w(x)x\} - Ft\varepsilon ph(x) \right] \\
= (1-t)\{ ph(x) - w(x)x \} + t\varepsilon ph(x)\{1-q(\varepsilon)F\},
\]
(4)
where \(q(\varepsilon)\) is the probability of being detected \((q > 0, q' > 0)\).

Substituting the equation (4) into (1), we have
\[
V = aph(x) + \beta \left[ (1-t)\{ ph(x) - w(x)x \} + t\varepsilon ph(x)\{1-q(\varepsilon)F\} \right].
\]
(5)
The objective \((V)\) is the function of \(x\) and \(\varepsilon\).

Setting the partial derivatives of (5) with respect to \(x\) or \(\varepsilon\) equal to zero, we get
\[
\frac{\partial V}{\partial x} = aph'(x) + \beta \left[ (1-t)\{ ph'(x) - w'(x)x - w(x) \} \right. \\
+ t\varepsilon ph'(x)\{1-q(\varepsilon)F\}] = 0,
\]
(6)
\[
\frac{\partial V}{\partial \varepsilon} = \beta tp h(x) \left\{ 1 - q(\epsilon) F - \varepsilon q' (\epsilon) F \right\} = 0.
\]
(7)
The second-order conditions require that
\[
v_{11} = \frac{\partial^2 V}{\partial x^2} < 0 \quad \text{and} \quad D = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} > 0,
\]
(8)
where \(v_{12}\) is the partial derivative of \(\partial V/\partial x\) with respect to \(\varepsilon\) and \(v_{22} = \partial^2 V/\partial \varepsilon^2\).

The second-order conditions are proved to be satisfied.\(^2\)

\(^2\) See Mathematical Appendix 1.
§ 3. The Effect of Raising the Tax Rate and the Penalty Rate

In this section, we examine the effect of raising the tax rate and the penalty rate on the quantity of labour employed ($x$) and the rate of the underreporting ($\varepsilon$).

Differentiating the first-order conditions (6) and (7) with respect to the tax rate yields:

\[
\begin{bmatrix}
  v_{11} & v_{12} \\
  v_{21} & v_{22}
\end{bmatrix}
\begin{bmatrix}
  \partial x / \partial t \\
  \partial \varepsilon / \partial t
\end{bmatrix}
= \begin{bmatrix}
  -v_{13} \\
  -v_{23}
\end{bmatrix},
\]

where $v_{13} = \partial^2 v / \partial x \partial t$ and $v_{23} = \partial^2 v / \partial \varepsilon \partial t$.

Therefore we obtain

\[
\frac{\partial x}{\partial t} = \left( -v_{13} v_{22} + v_{23} v_{12} \right) / D > 0,
\]

since $v_{12} = 0^3$, $v_{22} < 0^4$, $D > 0$ from (9), and

\[
v_{13} = \beta \left[ -\{ p h'(x) - w'(x)x - w(x) \} + \varepsilon p h'(x) (1 - q(\varepsilon) F) \right] > 0^5.
\]

In the same way, we obtain

\[
\frac{\partial \varepsilon}{\partial t} = \left( -v_{11} v_{23} + v_{21} v_{13} \right) / D = 0,
\]

since $v_{23} = \beta p h(x) \{ 1 - q(\varepsilon) F - \varepsilon q'(\varepsilon) F \} = 0$ from (7), and $v_{21} = 0^6$.

From (10), we come to a result that raising the tax rate will increase the quantity of labour employed, then increase the wage rate since the wage rate is the increasing function of the employment.\(^7\) On the other hand, from (11) raising the tax rate will not vary the rate of the underreporting.\(^8\)

Next, we will examine the effect of raising the penalty rate for the tax

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3) See Mathematical Appendix 1.
4) See Mathematical Appendix 1.
5) See Mathematical Appendix 2.
6) See Mathematical Appendix 1.
7) This is the same result as that in Watanabe (1989,a) which doesn't consider the sales revenue as a component of the objective.
8) This is also the same result as that in Watanabe (1989,a), which neglected the sales revenue as a component of the taxpayer's objective.
evasion. Differentiating the first-order conditions (6) and (7) with respect to the penalty rate $F$, and then making rearrangement, we get
\[
\begin{bmatrix}
v_{11} & v_{12} \\
v_{21} & v_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial F} \\
\frac{\partial \varepsilon}{\partial F}
\end{bmatrix}
=
\begin{bmatrix}
-v_{14} \\
-v_{24}
\end{bmatrix},
\]
where $v_{14} = \frac{\partial^2 v}{\partial x \partial F}$ and $v_{24} = \frac{\partial^2 v}{\partial \varepsilon \partial F}$.

Therefore, we get
\[
\frac{\partial x}{\partial F} = \frac{(-v_{14} v_{22} + v_{24} v_{12})}{D} < 0,
\]
(12) since $v_{22} < 0$, $D > 0$ from (9), $v_{12} = 0$, and $v_{14} = -\beta \varepsilon \rho h'(x) q(\varepsilon) < 0$ as $h'(x) > 0$.

In the same way, we obtain
\[
\frac{\partial \varepsilon}{\partial F} = \frac{(-v_{11} v_{24} + v_{21} v_{14})}{D} < 0,
\]
(13) since $v_{11} < 0$ from (8), $D > 0$ from (9), $v_{21} = 0$, and
\[v_{24} = -\beta \varepsilon \rho h(x) (q(\varepsilon) + \varepsilon q'(\varepsilon)) < 0\] as $q'(\varepsilon) > 0$.

From (12) and (13), we get the following result: raising the penalty rate decreases both the quantity of the employed labour and the rate of the underreporting.

In the following, we will examine the total effects of proportional changes in the tax rate and the penalty rate, which has not been examined in Watanabe (1989,a) .

Totally differentiating $x$ subject to $dF/F = dt/t$ and rearranging yield
\[
\frac{dx}{dF} \bigg|_{dF/F=dt/t} = \frac{\partial x}{\partial t} \frac{t}{F} + \frac{\partial x}{\partial F}.
\]
(14)

A sufficient condition for
\[
\frac{dx}{dF} \bigg|_{dF/F=dt/t} > 0,
\]
(15)
is

9) See Mathematical Appendix 1.

10) See Mathematical Appendix 1.

11) See Mathematical Appendix 1.

12) This is also the same result as that in Watanabe (1989,a).

13) See Mathematical Appendix 3.
$k \geq 1$,

where $k \equiv \frac{\varepsilon}{q} \frac{dq}{d\varepsilon}$ is the elasticity of $q(\varepsilon)$ with respect to $\varepsilon$.\(^{13}\)

In the same way, we have

$$\left. \frac{d\varepsilon}{dF} \right|_{dF/F=dt/t} = \frac{\partial \varepsilon}{\partial t} \frac{t}{F} + \frac{\partial \varepsilon}{\partial F} < 0, \quad (16)$$

since, from (11), (13), $\partial \varepsilon/\partial t = 0$ and $\partial \varepsilon/\partial F < 0$.

Therefore from (15), if the elasticity of $q(\varepsilon)$, [i.e. the probability of being detected,] with respect to $\varepsilon$, [i.e. the rate of underreporting,] is larger than or equal to 1, then the proportional changes in the tax rate and the penalty rate will increase the quantity of the employed labour, and from (16) the proportional changes will decrease the rate of underreporting at the same time.\(^{14}\)

Next, we will analyze additional problem concerning total tax collections.

Expected value of the total tax collections will be denoted as

$$T = (1-q(\varepsilon))t \{(1-\varepsilon)ph(x) - w(x)x\}$$

$$+ q(\varepsilon)\left[ t \{(1-\varepsilon)ph(x) - w(x)x\} + Fteph(x) \right]. \quad (17)$$

By taking partial differentiation of $T$ with respect to $F$, we can derive\(^ {15}\)

$$\frac{\partial T}{\partial F} > 0. \quad (18)$$

\(^{14}\) The effects of the proportional changes in the tax rate and the penalty rate on the quantity of the employed labour and the rate of underreporting have not been examined in Watanabe (1989,a) .

In Watanabe (1989,a) , without considering the sales revenue as a component of the objective, only the simple effect of raising the tax rate or the penalty rate has been analyzed.

In this paper and Watanabe (1989,a) the taxpayer is assumed to be a monopsonist. On the other hand, in Watanabe (1991,a) the case where the monopsonist is also a monopolist has been dealt with and the effects of the proportional changes in the tax rate and the penalty rate have also been examined, without considering the sales revenue as a component of the objective.

\(^{15}\) See Mathematical Appendix 4.
Therefore, from (18) raising the penalty rate will increase the expected value of the total tax collections.

On the other hand, the effect of raising the tax rate on the expected value of the total tax collections can not be determined unambiguously.\(^{16}\)

Therefore, in order to increase the expected value of the total tax collections unambiguously, raising the penalty rate will be superior to raising the tax rate.

§ 4. Further Effects of Considering the Sales Revenue

In this section, first we examine the effects of considering the sales revenue as a component of the taxpayer's objective on the quantity of the employed labour \((x)\) and the rate of the underreporting \((\epsilon)\).

Differentiating the first-order conditions (6) and (7) with respect to \(\alpha\) yields.

\[
\begin{bmatrix}
v_{11} & v_{12} \\
v_{21} & v_{22}
\end{bmatrix}
\begin{bmatrix}
\partial x / \partial \alpha \\
\partial \epsilon / \partial \alpha
\end{bmatrix} =
\begin{bmatrix}
-v_{15} \\
-v_{25}
\end{bmatrix},
\]

where \(v_{15} = \partial^2 v / \partial x \partial \alpha\) and \(v_{25} = \partial^2 v / \partial \epsilon \partial \alpha\).

Hence, we obtain

\[
\frac{\partial x}{\partial \alpha} = \left( -v_{15}v_{22} + v_{25}v_{12} \right) / D > 0, \tag{19}
\]

since \(v_{12} = 0^{17}, \quad v_{22} < 0^{18}, \quad D > 0\) from (9) and \(v_{15} > 0.\(^{19}\)

In the same way, we get

\[
\frac{\partial \epsilon}{\partial \alpha} = \left( -v_{11}v_{25} + v_{21}v_{15} \right) / D = 0, \tag{20}
\]

since \(v_{25} = 0\) from (7) and \(v_{21} = 0.\(^{20}\)

Therefore, from (19) and (20), if the taxpayer's objective depends further on the sales revenue, then the quantity of the employed labour will

\[\text{\textsuperscript{16}}\text{ See Mathematical Appendix 5.}\]
\[\text{\textsuperscript{17}}\text{ See Mathematical Appendix 1.}\]
\[\text{\textsuperscript{18}}\text{ See Mathematical Appendix 1.}\]
\[\text{\textsuperscript{19}}\text{ See Mathematical Appendix 6.}\]
\[\text{\textsuperscript{20}}\text{ See Mathematical Appendix 1.}\]
be increased, though the rate of underreporting will not be affected.

§ 5. Declared Sales Revenue instead of True Sales Revenue

It is also plausible that a declared sales revenue instead of the true sales revenue is considered as a component of the taxpayer’s objective. When the declared sales revenue is considered as a component of the objective, the objective shown by (1) will be modified in the following manner.

\[ \tilde{V} = a (1 - \tilde{\varepsilon}) \tilde{p} h(\tilde{x}) + \beta \tilde{E}, \]

(21)

Where \( \tilde{x} \), for example, corresponds to \( x \) in the preceding sections.

Therefore, in the same way, we obtain the modified objective,

\[ \tilde{V} = a (1 - \tilde{\varepsilon}) \tilde{p} h(\tilde{x}) \]

\[ + \beta [(1 - t) \{ \tilde{p} h(x) - w(\tilde{x}) \tilde{x} \} + t \tilde{\varepsilon} \tilde{p} h(\tilde{x}) \{ 1 - q(\tilde{\varepsilon}) F \}]. \]

The objective (\( \tilde{V} \)) is the function of \( \tilde{x} \) and \( \tilde{\varepsilon} \).

Setting the partial derivatives of (22) with respect to \( \tilde{x} \) or \( \tilde{\varepsilon} \) equal to zero, we have

\[ \frac{\partial \tilde{V}}{\partial \tilde{x}} = a (1 - \tilde{\varepsilon}) \tilde{p} h'(\tilde{x}) + \beta [(1 - t) \{ \tilde{p} h'(x) - w'(\tilde{x}) \tilde{x} - w(\tilde{x}) \} + t \tilde{\varepsilon} \tilde{p} h'(\tilde{x}) \{ 1 - q(\tilde{\varepsilon}) F \}] = 0, \]

(23)

\[ \frac{\partial \tilde{V}}{\partial \tilde{\varepsilon}} = -a \tilde{p} h(\tilde{x}) + \beta t \tilde{p} h(\tilde{x}) \{ 1 - q(\tilde{\varepsilon}) F - \tilde{\varepsilon} q'(\tilde{\varepsilon}) F \} = 0. \]

(24)

The second-order conditions require that

\[ \tilde{v}_{11} = \frac{\partial^2 \tilde{V}}{\partial \tilde{x}^2} < 0, \]

(25)

and

\[ \tilde{D} = \begin{vmatrix} \tilde{v}_{11} & \tilde{v}_{12} \\ \tilde{v}_{21} & \tilde{v}_{22} \end{vmatrix} > 0. \]

(26)

The second-order conditions are proved to be satisfied.\(^{21}\)

Differentiating the first-order conditions (23) and (24) with respect to the tax rate yields.

---

21) See Mathematical Appendix 7.
Sales Revenue and Expected Profit of Monopsonist, Vanishing of Discontinuous Segment, & Tax Evasion.

\[
\begin{bmatrix}
\tilde{v}_{11} & \tilde{v}_{12} \\
\tilde{v}_{21} & \tilde{v}_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial t} \\
\frac{\partial \tilde{e}}{\partial t}
\end{bmatrix} = 
\begin{bmatrix}
-\tilde{v}_{13} \\
-\tilde{v}_{23}
\end{bmatrix},
\]

where \( \tilde{v}_{13} = \frac{\partial^2 \tilde{v}}{\partial \tilde{x} \partial t} \) and \( \tilde{v}_{23} = \frac{\partial^2 \tilde{v}}{\partial \tilde{e} \partial t} \).

Hence, we get

\[
\frac{\partial \tilde{v}}{\partial t} = (-\tilde{v}_{13} \tilde{v}_{22} + \tilde{v}_{23} \tilde{v}_{12}) / \tilde{D} > 0,
\]

since \( \tilde{v}_{12} = 0^{22} \), \( \tilde{v}_{22} < 0^{23} \), \( \tilde{D} > 0 \) from (26) and

\[
\tilde{v}_{13} = \beta \left[ -\{ \rho \tilde{h}'(\tilde{x}) - w'(\tilde{x}) \tilde{x} - w(\tilde{x}) \} + \tilde{e} \rho \tilde{h}'(\tilde{x}) \{ 1 - q(\tilde{e})F \} \right] > 0.24
\]

In the same way, we obtain

\[
\frac{\partial \tilde{e}}{\partial t} = (-\tilde{v}_{11} \tilde{v}_{23} + \tilde{v}_{21} \tilde{v}_{13}) / \tilde{D} > 0,
\]

since \( \tilde{v}_{23} = \beta \rho \tilde{h}(\tilde{x}) \{ 1 - q(\tilde{e})F \} - \epsilon \tilde{e} \tilde{q}'(\tilde{e})F > 0 \), \( \tilde{v}_{21} = 0 \), \( \tilde{v}_{11} < 0 \) from (25) and \( \tilde{D} > 0 \) from (26).

From (27) and (28), we come to a result that raising the tax rate will increase both the quantity of the employed labour and the rate of the underreporting. Though the result with respect to the quantity of the employed labour derived from (27) is the same as that in the preceding sections, the result with respect to the rate of the underreporting derived from (28) is different from that in the preceding sections.

Next, we will examine the effects of raising the penalty rate.

Differentiating the first-order conditions (23) and (24) with respect to the penalty rate yields

\[
\begin{bmatrix}
\tilde{v}_{11} & \tilde{v}_{12} \\
\tilde{v}_{21} & \tilde{v}_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial F} \\
\frac{\partial \tilde{e}}{\partial F}
\end{bmatrix} = 
\begin{bmatrix}
-\tilde{v}_{14} \\
-\tilde{v}_{24}
\end{bmatrix},
\]

where \( \tilde{v}_{14} = \frac{\partial^2 \tilde{v}}{\partial \tilde{x} \partial F} \) and \( \tilde{v}_{24} = \frac{\partial^2 \tilde{v}}{\partial \tilde{e} \partial F} \).

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22) See Mathematical Appendix 7.
23) See Mathematical Appendix 7.
24) See Mathematical Appendix 8.
25) See Mathematical Appendix 9.
26) See Mathematical Appendix 7.
27) See Mathematical Appendix 7.
28) See Mathematical Appendix 7.
Hence, we obtain
\[
\frac{\partial \tilde{x}}{\partial F} = \left( -\tilde{v}_{14} \tilde{v}_{22} + \tilde{v}_{24} \tilde{v}_{12} \right) / \tilde{D} < 0, \tag{29}
\]
since \( \tilde{v}_{22} < 0 \)\textsuperscript{27}, \( \tilde{v}_{12} = 0 \)\textsuperscript{28}, \( \tilde{D} > 0 \) from (26), and
\( \tilde{v}_{14} = -\beta t \tilde{\varepsilon} \tilde{p} K(\tilde{x}) q(\tilde{\varepsilon}) < 0. \)

In the same way, we get
\[
\frac{\partial \tilde{\varepsilon}}{\partial F} = \left( -\tilde{v}_{11} \tilde{v}_{24} + \tilde{v}_{21} \tilde{v}_{14} \right) / \tilde{D} < 0, \tag{30}
\]
since \( \tilde{v}_{11} < 0 \) from (25), \( \tilde{v}_{21} = 0 \)\textsuperscript{29}, \( \tilde{D} > 0 \) from (26), and
\( \tilde{v}_{24} = -\beta t \tilde{p} \tilde{h}(\tilde{x}) \left\{ q(\tilde{\varepsilon}) + \tilde{\varepsilon} q'(\tilde{\varepsilon}) \right\} < 0 \) as \( q'(\tilde{\varepsilon}) > 0. \)

From (29) and (30), we come to a result that raising the penalty rate will decrease both the quantity of the employed labour and the rate of the underreporting. These results are the same as those in the preceding sections.

In the following, we will examine the total effects of proportional changes in the tax rate and the penalty rate.

Totally differentiating \( \tilde{x} \) subject to \( dF/F = dt/t \) and rearranging yield
\[
\frac{dx}{dF} \bigg|_{dF/F=dt/t} = \frac{\partial x}{\partial t} \frac{t}{F} + \frac{\partial x}{\partial F}. \tag{31}
\]

A sufficient condition for
\[
\frac{dx}{dF} \bigg|_{dF/F=dt/t} > 0, \tag{32}
\]
is
\( \tilde{k} \geq 1. \)

where \( \tilde{k} = \frac{\tilde{\varepsilon}}{q \frac{dq}{d\tilde{\varepsilon}}} \) is the elasticity of \( q(\tilde{\varepsilon}) \) with respect to \( \tilde{\varepsilon} \)\textsuperscript{30}.

Hence, if the elasticity of the probability to be detected with respect to the rate of underreporting is larger than or equal to one, the quantity of the employed labour will be increased by raising both the tax rate and the penalty rate proportionally. This result in this section is the same as that

\textsuperscript{29) See Mathematical Appendix 7.}
\textsuperscript{30) See Mathematical Appendix 10.}
in the preceding sections.

On the other hand, with respect to the rate of the underreporting, we have

$$\left. \frac{d\tilde{e}}{dF} \right|_{dF/F = dt/t} = \frac{\partial \tilde{e}}{\partial t} \frac{t}{F} + \frac{\partial \tilde{e}}{\partial F}.$$  \hspace{1cm} (33)

A sufficient condition for \( \left. \frac{d\tilde{e}}{dF} \right|_{dF/F = dt/t} < 0 \), can be derived\(^{31}\) from (28), (30) and (33) as

$$\frac{1}{2} t > \frac{a}{\beta}.$$ \hspace{1cm} (34)

This result is different from that in the preceding sections, since (16) doesn't require the condition (34).

If the half of the tax rate is larger than ratio between \( a \) and \( \beta \), then the rate of underreporting will be decreased by raising both the tax rate and the penalty rate proportionally.

Next, the total tax collections will be analyzed in the following.

Expected value of the total tax collections will be denoted as

$$\tilde{T} = (1 - q(\tilde{e})) t \{ (1 - \tilde{e}) p h(\tilde{x}) - w(\tilde{x}) \tilde{x} \}$$

$$+ q(\tilde{e}) [ t \{ (1 - \tilde{e}) p h(\tilde{x}) - w(\tilde{x}) \tilde{x} \} + F t \tilde{e} p h(\tilde{x}) ]$$ \hspace{1cm} (35)

By taking partial differentiation of \( \tilde{T} \) with respect to \( F \), we get\(^{32}\)

$$\frac{\partial \tilde{T}}{\partial F} > 0.$$ \hspace{1cm} (36)

Hence, raising the penalty rate will increase the expected value of the total tax collections. This is the same result as that of (18).

On the other hand, the effect of raising the tax rate on the expected value of the total tax collections cannot be determined.\(^{33}\)

This is also the same result as that of section 3. Therefore we come to a result that in order to increase the expected value of the total tax

\(^{31}\) See Mathematical Appendix 11.

\(^{32}\) See Mathematical Appendix 12.

\(^{33}\) See Mathematical Appendix 13.
collections unambiguously, raising the penalty rate will be superior to raising the tax rate, whether the declared sales revenue is considered, instead of the true sales revenue, as a component of the taxpayer's objective or not.

Next, we will examine the effects of considering the sales revenue as a component of the taxpayer's objective on the quantity of the employed labour and the rate of the underreporting.

Differentiating the first-order conditions (23) and (24) with respect to \( \alpha \) yields\(^{34)}\)

\[
\frac{\partial x}{\partial \alpha} > 0. \tag{37}
\]

This is the same result as that of (19). Hence, if the taxpayer's objective depends further on the sales revenue, then the quantity of the employed labour will be increased, whether the declared sales revenue is considered, instead of the true sales revenue, as a component of the taxpayer's objective or not.

In the same way, we have\(^{35)}\)

\[
\frac{\partial \varepsilon}{\partial \alpha} < 0. \tag{38}
\]

This is a different result from that of (20). Therefore, from (38) if the taxpayer's objective depends further on the sales revenue, then the rate of underreporting will be reduced, when the declared sales revenue is considered as a component of the taxpayer's objective. On the other hand, when the true sales revenue is considered as a component of the taxpayer's objective, the rate of underreporting will not be affected by raising \( \alpha \) from (20).

Results derived from the analyses of this section and preceding sections will be summarized in the following tables.

\(^{34)}\) See Mathematical Appendix 14.

\(^{35)}\) See Mathematical Appendix 15.
### Table 1: True & Declared Sales Revenue

<table>
<thead>
<tr>
<th></th>
<th>true sales revenue as a component of objective</th>
<th>declared sales revenue as a component of objective</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$: quantity of employed labour</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td>$\varepsilon$: rate of underreporting</td>
<td>[0]</td>
<td>[+]</td>
</tr>
<tr>
<td>$\tilde{x}$: quantity of employed labour</td>
<td>(-)</td>
<td>(-)</td>
</tr>
<tr>
<td>$\tilde{\varepsilon}$: rate of underreporting</td>
<td>(-)</td>
<td>(-)</td>
</tr>
<tr>
<td>$t$: tax rate</td>
<td>(−)</td>
<td>(−)</td>
</tr>
<tr>
<td>$F$: penalty rate</td>
<td>(-)</td>
<td>(-)</td>
</tr>
<tr>
<td>$dt / t = dF / F$: proportional changes</td>
<td>(+) *</td>
<td>(+) **</td>
</tr>
</tbody>
</table>

* $k \equiv \frac{\varepsilon d \varphi}{q d \varepsilon} \geq 1$ is a sufficient condition for this sign.

** $\tilde{k} \equiv \frac{\varepsilon d \varphi}{q d \varepsilon} \geq 1$ is a sufficient condition for this sign.

*** $\frac{1}{2} t > \frac{a}{\beta}$ is a sufficient condition for this sign.

### Table 2: Effects of $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$x$</th>
<th>$\varepsilon$</th>
<th>$\tilde{x}$</th>
<th>$\tilde{\varepsilon}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[+]</td>
<td>[0]</td>
<td>(+)</td>
<td>(-)</td>
<td></td>
</tr>
</tbody>
</table>

### Table 3: Effects on the Expected Value of Total tax collections

<table>
<thead>
<tr>
<th></th>
<th>Expected Value of Total tax collections</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T$: the case of true sales revenue</td>
</tr>
<tr>
<td>$t$: tax rate</td>
<td>(?)</td>
</tr>
<tr>
<td>$F$: penalty rate</td>
<td>(+)</td>
</tr>
</tbody>
</table>
§ 6. Vanishing of Discontinuous Segment

The purpose of this section is to show that even if a demand curve is kinked, a discontinuous segment (as a cause of) price rigidity may vanish when a tax evasion is taken into consideration.

To show the possibility of vanishing of the discontinuous segment, we will analyze a more plausible model than Watanabe (1991,b).

From the analysis of Watanabe (1991,b), we have obtained several results under an implicit assumption that if one of the duopolists (I) reduces his price, his rival (II) will react by increasing his output to

36) Since the pioneering works by Sweezy (1939) and Hall and Hitch (1939), the kinked-demand-curve analysis has been well known. See, for example, Imai, Uzawa, Komiya, Negishi and Murakami (1971).

37) Since Allingham and Sandmo (1972), the tax evasion has been analyzed in various papers (for example, Yitzhaki (1974), Watanabe (1986,a, 1987, 1988,a, 1989,a, 1989,b, 1990,a, 1991,a, 1991,b)).

38) We must correct one of the several results. In Watanabe (1991,b) we had one incorrect result, that is, raising the tax will reduce the vertical discontinuous segment, which will implicate to raise the possibility of the price changes due to the changes of the marginal cost. It is correct that raising the tax rate will increase the length of the vertical discontinuous segment (where slight variations in the marginal cost (α) will not change the price), which will implicate to reduce the possibility of the price changes due to the changes of the marginal cost. As derived by Watanabe (1991,b) it is also correct that raising the tax rate will reduce the length of the vertical discontinuous segment [denoted as VS in Watanabe (1991,b)] which allows slight changes in (1-t)α without changing the price, where t is the tax rate and α is the marginal cost. However, reducing VS by raising the tax rate is not a sufficient condition for reducing the length of vertical discontinuous segment concerning the variations in α (marginal cost), which will be denoted by VS/(1-t).

Rather, raising the tax rate will increase the length of the vertical discontinuous segment denoted by VS/(1-t) which concerns the variations in the marginal cost (α), since ∂[VS / 1-t] / ∂t > 0 can be obtained by differentiation.

On the other hand, concerning the effect of raising the penalty rate we had a correct result since ∂VS / ∂F < 0 is straightforwardly a sufficient condition for ∂[VS / 1-t] / ∂F < 0.
attain $Q_2 = Q_1$, where $Q_1$ is the output level of (I) and $Q_2$ is the output level of his rival (II). However, it will be more plausible to assume that his rival (II) will react by increasing his output to attain $Q_2 = (1 - \varepsilon)Q_1$, which is the declared output level of (I).

In the following, the same functions and the same numerical example except the cost functions as Henderson and Quandt (1971) will be used. And to make the analysis simple, we assume that the rate ($\varepsilon$) of under-reporting gross revenues is a parameter in this section. In this section, further, as stated in the introduction, the sales revenue is not considered as a component of the taxpayer's objective for simplicity.

According to Henderson and Quandt (1971), we have
\[
P_1 = 100 - 2Q_1 - Q_2, \\
P_2 = 95 - Q_1 - 3Q_2, \\
C_1 = \alpha Q_1, \\
C_2 = \beta Q_2,
\]
where $P_1$, $P_2$, $Q_1$ and $Q_2$ are the prices, the output levels of the duopolists and $\alpha > 0$, $\beta > 0$. Further, as Henderson and Quanot (1971), we assume that the predetermined prices and quantities are $P_1 = 70$, $Q_1 = 10$, $P_2 = 55$, and $Q_2 = 10$.

If the tax evasion by underreporting gross revenues is not detected, the profit will be
\[
P_1Q_1 - C_1 - t \left[ (1 - \varepsilon)P_1Q_1 - C_1 \right], \quad (39)
\]
where $\varepsilon$ is the rate of under-reporting gross revenues. On the other hand, if the tax evasion is detected, the profit will be
\[
P_1Q_1 - C_1 - t \left[ (1 - \varepsilon)P_1Q_1 - C_1 \right] - F \varepsilon P_1Q_1, \quad (40)
\]
where $F$ is the penalty rate for the tax evasion.

Hence, from (39) and (40) the expected profit ($E \pi$) is
\[
E \pi = P_1Q_1 - C_1 - t \left[ (1 - \varepsilon)P_1Q_1 - C_1 \right] - q(\varepsilon)F \varepsilon P_1Q_1, \quad (41)
\]
where $q(\varepsilon)$ [$q'(\varepsilon) > 0$, $q''(\varepsilon) > 0$] is the probability of the tax evasion being detected.
If one of the duopolists (I) raised his price, his rival (II) would leave his own price unchanged at 55 dollars. Hence, the expected profit given by (41) becomes
\[ E\pi = \{1-t(1-\epsilon)-q(\epsilon)Ft\epsilon\} \frac{260-5Q_1}{3} Q_1 - a(1-t)Q_1. \] (42)

The first term of (42) corresponds to a total revenue and the second term of (42) corresponds to a total cost in the case of raising the price, when the tax is not considered.

On the other hand, if one of the duopolists (I) reduces his price, his rival will react by increasing his output to attain \( Q_2 = (1-\epsilon)Q_1 \), which is the declared output level of (I).

Hence, in this case, the expected profit given by (41) becomes 39)
\[ E\pi = \{1-t(1-\epsilon)-q(\epsilon)Ft\epsilon\} \{100-(3-\epsilon)Q_1\} Q_1 \] (43)
\[ - a(1-t)Q_1. \]

The first term of (43) corresponds to a total revenue and the second term of (43) corresponds to a total cost in the case of reducing the price, when the tax is not considered.

Differentiating the first term of (42) with respect to \( Q_i \), we have
\[ \frac{1}{3} \{1-t(1-\epsilon)-q(\epsilon)Ft\epsilon\}(260-10Q_1), \] (44)

Differentiating the second term \([a(1-t)Q_1]\) of (42) with respect \( Q_i \), we get
\[ (1-t) a, \] (45)
where \( a \) is a marginal cost.

On the other hand, differentiating the first term of (43) with respect to \( Q_i \), we have
\[ \{1-t(1-\epsilon)-q(\epsilon)Ft\epsilon\}\{100-2(3-\epsilon)Q_1\}. \] (46)
Differentiating the second term of (43) with respect to \( Q_i \), we get
\[ (1-t) a, \]

---

39) Substituting \( Q_2 = (1-\epsilon)Q_1 \) into the demand function: \( P_i = 100-2Q_i-Q_2 \), we get \( P_1 = 100-(3-\epsilon)Q_i \).
which is equal to (45).

As stated above, the rate \( \varepsilon \) of underreporting is assumed to be a parameter in this section.

If we don't take the tax evasion into consideration, the upper bound of \((1-t) a\), where \( a \) is the marginal cost, for keeping the price unchanged will be \((1-t) \frac{160}{3}\) from (44) since the predetermined \( Q \), is 10.

And the lower bound of \((1-t) a\) for keeping the price unchanged will be \((1-t) 40\) from (46) if we don't take the tax evasion into consideration.

Therefore so long as
\[
(1-t) \frac{160}{3} \geq (1-t) a \geq (1-t) 40
\]
or \[
\frac{160}{3} \geq a \geq 40,
\] (47)
the price will be kept unchanged.

However, if we take the tax evasion into consideration, the upper bound of the marginal cost \((a)\) for keeping the price unchanged will be
\[
\frac{160 \{1-t(1-\varepsilon) - q(\varepsilon) Ft\varepsilon\}}{3(1-t)},
\]
from (44).

The lower bound of the marginal cost \((a)\) for keeping the price unchanged will be
\[
\frac{(40+20\varepsilon) \{1-t(1-\varepsilon) - q(\varepsilon) Ft\varepsilon\}}{(1-t)},
\]
from (46).

Therefore, so long as
\[
\frac{160 \{1-t(1-\varepsilon) - q(\varepsilon) Ft\varepsilon\}}{3(1-t)} \geq a \geq \frac{(40+20\varepsilon) \{1-t(1-\varepsilon) - q(\varepsilon) Ft\varepsilon\}}{(1-t)},
\] (48)
the price will not be changed, where \( a \) is the marginal cost, (See Fig. 1).

When the tax evasion is not considered, i.e. \( \varepsilon \) is assumed to be zero, the inequalities (48) will be reduced to the inequalities (47). Hence, when the tax evasion is not considered, the length of the discontinuous segment will be
\[
\frac{160}{3} - 40 = \frac{40}{3},
\]

since \( 1-t(1-\varepsilon) - q(\varepsilon)Ft\varepsilon \) is \( 1-t \) if \( \varepsilon = 0 \).

On the other hand, when the tax evasion is taken into consideration, the length of the discontinuous segment will be
\[
\frac{1-t(1-\varepsilon) - q(\varepsilon)Ft\varepsilon}{(1-t)} \left( \frac{40}{3} - 20\varepsilon \right),
\]

from (48).

To make the slope of the marginal revenue function in Fig 1 negative, \( 1-t(1-\varepsilon) - q(\varepsilon)Ft\varepsilon \) will be assumed to be positive.

Hence, if \( \varepsilon < 2/3 \), then the length of the discontinuous segment will be positive. However, if \( \varepsilon = 2/3 \), then the discontinuous segment will vanish.\(^{40}\)

\(^{40}\) In general, we cannot exclude the possibility of \( \varepsilon > 2/3 \).
§ 7. Concluding Remarks

The first purpose of this paper is to consider both the sales revenue and the expected profit as the objective of the taxpayer, because the aspect of sales revenue has been neglected in the context of the tax evasion. Not only the expected profit but also the sales revenue will be important components of the taxpayer's objective.

We have examined two cases; (1) true sales revenue is a component of the taxpayer's objective, (2) declared sales revenue is a component of the taxpayer's objective.

Main results of the first purpose of this paper will be summarized in the following. From table 1, we come to a result that raising the tax rate will increase the rate of underreporting for the tax evasion if the taxpayer's objective depends on the declared sales revenue as well as the expected profit. On the other hand, if the taxpayer's objective depends on the true sales revenue as well as the expected profit, raising the tax rate will not very the rate of underreporting. From table 2, we get a result that if the taxpayer's objective depends further (i.e. $\alpha$ is increased) on the sales revenue, the rate of underreporting will be reduced only in the case where the declared sales revenue is considered.

From table 3, we also obtain a result that raising the penalty rate will be superior to raising the tax rate in order to increase the expected value of the total tax collections which include not only the regular tax revenue but also the tax revenue due to the penalty when the tax evasion is detected.

Main result of the another purpose of this paper will be summarized in the following.

We come to a somewhat surprising result that even if the demand curve is kinked, the discontinuous segment (which allows marginal cost to change without price changes,) may vanish when the tax evasion is taken
into consideration. The discontinuous segment has long been regarded as one of the causes of price rigidity. However, if we cannot exclude the possibility of vanishing of the discontinuous segment, even the slight variations in the marginal cost may induce the price change. As a matter of fact, prices are not necessarily rigid as pointed out, for example, by Stigler (1947).

Mathematical Appendix 1.

\[ v_{11} = a p h'(x) + \beta \left[ (1-t) \{ p h'(x) - w'(x)x - w(x) - w'(x) \} \right] + t e p h'(x) \{ 1 - q(\epsilon) F \} < 0, \]

since, \( 1 - q(\epsilon) F = e q(\epsilon) F > 0 \) from the equation (7), and \( h'(x) < 0 \) is assumed, which means the decreasing marginal productivity of the labour and \( w'(x) > 0, w''(x) > 0 \) are assumed with respect to the wage rate.

\[ D = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} = v_{11}v_{22} > 0, \]

since, \( v_{12} = \beta t p h'(x) \{ 1 - q(\epsilon) F - e q(\epsilon) F \} \) from the equation (6) and \( v_{12} = 0 \) from the equation (7), \( v_{11} < 0 \) from (8) and \( v_{22} = \beta t p h(x) \{ -q(\epsilon) F - e q(\epsilon) F - e q'(\epsilon) F \} < 0 \) as \( q'(\epsilon) > 0 \) is assumed, which means that the probability of being detected is an increasing function of \( \epsilon \) and \( q'(\epsilon) > 0 \) is also assumed.

Mathematical Appendix 2.

\[ p h'(x) - w'(x)x - w(x) = - \left[ a p h'(x) + \beta t e p h'(x) \{ 1 - q(\epsilon) F \} \right] / (1 - t)(1 - t), \]

from (6). \( 1 - q(\epsilon) F = e q(\epsilon) F \) from (7). Then, \( 1 - q(\epsilon) F > 0 \), since \( q'(\epsilon) > 0 \). Hence, \( p h'(x) - w'(x)x - w(x) < 0 \).

Therefore, \( v_{13} > 0 \).

Mathematical Appendix 3.

Using (10) and (12), the equation 14 is rewritten as
\[
\frac{dx}{dF} \bigg|_{dF/dt} = \left\{ v_{13} (t/F) + v_{14} \right\} / v_{11}
\]
\[
= - \frac{t \beta}{v_{11} F} \left[ \{ p \, h'(x) - w'(x) x - w(x) \} + \varepsilon \rho \, h'(x) \{ 1 - q(\varepsilon) F \} \right]
\]
\[
+ \frac{1}{v_{11}} \beta t \varepsilon \rho \, h'(x) q(\varepsilon).
\]

From (7), \(1 - q(\varepsilon) F = \varepsilon q(\varepsilon) F \). Then,
\[
\frac{dx}{dF} \bigg|_{dF/dt} = \frac{t \beta}{v_{11} F} \{ p \, h'(x) - w'(x) x - w(x) \}
\]
\[
- \frac{\beta \varepsilon \rho \, h'(x) q(\varepsilon) (k-1) t}{v_{11}}.
\]

From Mathematical Appendix 2, \( p \, h'(x) - w'(x) x - w(x) < 0 \), and \( v_{11} < 0 \) from (8). Therefore \( k \geq 1 \) is a sufficient condition for (15).

Mathematical Appendix 4.
\[
T = t \{ p \, h(x) - w(x) x \} - t \varepsilon \rho \, h(x) \{ 1 - q(\varepsilon) F \}
\]
\[
\frac{\partial T}{\partial F} = t \{ p \, h'(x) - w'(x) x - w(x) \} \frac{\partial x}{\partial F}
\]
\[
- t \varepsilon \rho \, h'(x) \{ 1 - q(\varepsilon) F \} \frac{\partial x}{\partial F}
\]
\[
+ t \varepsilon \rho \, h(x) \, q(\varepsilon) \{ 1 - q(\varepsilon) F - \varepsilon q(\varepsilon) F \} \rho \, h(x) \frac{\partial \varepsilon}{\partial F} > 0,
\]

since \( p \, h'(x) - w'(x) x - w(x) < 0 \) from Mathematical Appendix 2, \( \partial x / \partial F < 0 \) from (12), \( 1 - q(\varepsilon) F - \varepsilon q(\varepsilon) F = 0 \), then \( 1 - q(\varepsilon) F = \varepsilon q(\varepsilon) F > 0 \) from (7), and \( \partial x / \partial t > 0 \) from (10).
Mathematical Appendix 5.

\[ \frac{\partial T}{\partial t} = ph(x) - w(x)x + t \{ p h'(x) - w'(x)x - w(x) \} \frac{\partial x}{\partial t} \]

\[ -\epsilon ph(x) \{ 1 - q(\epsilon)F \} \]

\[ -t\epsilon ph'(x) \{ 1 - q(\epsilon)F \} \frac{\partial x}{\partial t} \]

\[ = (1 - \epsilon) ph(x) + \epsilon ph(x)q(\epsilon)F - w(x)x \]

\[ + t \{ p h'(x) - w'(x)x - w(x) \} \frac{\partial x}{\partial t} \]

\[ -t\epsilon ph'(x) \{ 1 - q(\epsilon)F \} \frac{\partial x}{\partial t} \]

\[ = (1 - \epsilon + \epsilon q(\epsilon)F) ph(x) - w(x)x \]

\[ + t \{ p h'(x) - w'(x)x - w(x) \} \frac{\partial x}{\partial t} \]

\[ -t\epsilon ph'(x) \{ 1 - q(\epsilon)F \} \frac{\partial x}{\partial t} . \]

The first term is positive and other terms are negative since \( p h'(x) - w'(x)x - w(x) < 0 \) from Mathematical Appendix 2, \( \partial x/\partial t > 0 \) from (10) and \( 1 - q(\epsilon)F = \epsilon q'(\epsilon)F > 0 \) from (7). Then, the sign of \( \partial T/\partial t \) can not be determined unambiguously.

Mathematical Appendix 6.

\[ v_1 = \alpha ph'(x) + \beta \{ (1 - t) \{ p h'(x) - w'(x)x - w(x) \} + t\epsilon ph'(x) \{ 1 - q(\epsilon)F \} \} \]

Then, \( v_{15} = \partial^2 v/\partial x \partial \alpha = ph'(x) > 0. \)

Mathematical Appendix 7.

\[ \tilde{v}_{11} = \alpha (1 - \epsilon) p h''(\tilde{x}) \]

\[ + \beta \{ (1 - t) \{ p h''(\tilde{x}) - w''(\tilde{x})\tilde{x} - w'(\tilde{x}) - w'(\tilde{x}) \} + t\epsilon ph''(\tilde{x}) \{ 1 - q(\tilde{\epsilon})F \} \} \]

\[ < 0, \]

since \( h''(\tilde{x}) < 0, w''(\tilde{x}) > 0, w'(\tilde{x}) > 0 \) and \( 1 - q(\tilde{\epsilon})F = \frac{\alpha}{\beta t} + \tilde{\epsilon} q'(\tilde{\epsilon})F > 0 \) from (24).

\[ \tilde{D} = \tilde{v}_{11} \tilde{v}_{22} - \tilde{v}_{12}^2 > 0, \]
since $v_{11} < 0$,

$$v_{22} - \frac{\partial^2 v}{\partial \epsilon^2} = \beta t p h(\epsilon)(-q'(\epsilon)F - q'(\epsilon)F - \epsilon q'(\epsilon)F) < 0$$

as $q'(\epsilon) > 0$ and $\epsilon > 0$, and $v_{12} = p h'(\epsilon)[-\alpha + \beta t(1 - q(\epsilon)F - \epsilon q'(\epsilon)F)] = 0$, since

$$-\alpha + \beta t(1 - q(\epsilon)F - \epsilon q'(\epsilon)F) = 0$$

from (24).

Mathematical Appendix 8.

From (23) we get

$$\{p h'(\tilde{\epsilon}) - w'(\tilde{\epsilon})\tilde{x} - w(\tilde{\epsilon})\}
= -\frac{a}{\beta(1-t)} p h'(\tilde{\epsilon}) - \frac{t}{1-t} \epsilon p h'(\tilde{\epsilon})\{1 - q(\tilde{\epsilon})F\} < 0$$

since $h'(\tilde{\epsilon}) > 0$ and $1 - q(\tilde{\epsilon})F > 0$ from Mathematical Appendix 7. Hence, $\tilde{v}_{13} > 0$.

Mathematical Appendix 9.

From (24) we get

$$1 - q(\tilde{\epsilon})F - \epsilon q'(\tilde{\epsilon})F = \frac{a}{\beta t} > 0.$$

Mathematical Appendix 10.

$$\frac{d\tilde{x}}{dF}\bigg|_{dF/dt = \frac{1}{t}} = \frac{\partial \tilde{x}}{\partial t} \cdot \frac{1}{F} + \frac{\partial \tilde{x}}{\partial F}$$

$$= -\frac{\beta}{v_{11}} \left[\{p h'(\tilde{x}) - w'(\tilde{x})\tilde{x} - w(\tilde{x})\} + \epsilon p h'(\tilde{x})\{1 - q(\epsilon)F\}\right] + \frac{\beta}{v_{11}} \epsilon p h'(\tilde{x})q(\epsilon).$$

From (24), $1 - q(\tilde{\epsilon})F = \frac{a}{\beta t} + \epsilon q'(\tilde{\epsilon})F$.

Hence,

$$\frac{d\tilde{x}}{dF}\bigg|_{dF/dt = \frac{1}{t}} = \frac{\beta}{v_{11}} \left[p h'(\tilde{x}) - w'(\tilde{x})\tilde{x} - w(\tilde{x})\right] + \frac{\beta}{v_{11}} \epsilon p h'(\tilde{x})q(\epsilon)\left\{-\frac{a}{\beta F t q(\epsilon)}\tilde{k} + 1\right\}.$$
where \( \tilde{k} \equiv \frac{\tilde{e}}{q(\tilde{e})} q'(\tilde{e}) \).

Therefore, \( k \geq 1 \) is a sufficient condition for \( \frac{dx}{dF} \bigg|_{dF/dt=1} > 0 \), since

\[ ph'(\tilde{x}) - w'(\tilde{x}) \tilde{x} - w(\tilde{x}) < 0 \]

from Mathematical Appendix 8, \( \tilde{v}_{11} < 0 \) from (25) and \( h'(\tilde{x}) > 0 \).

Mathematical Appendix 11.

From (28), (30) and (33), we get

\[ \frac{d\tilde{e}}{dF} \bigg|_{dF/dt=1} = \frac{1}{\tilde{v}_{22}} \beta t ph(\tilde{x}) \frac{1}{F} \left[ -1 + 2F \{ q(\tilde{e}) + \tilde{e} q'(\tilde{e}) \} \right] \]

From (24), \( F \{ q(\tilde{e}) + \tilde{e} q'(\tilde{e}) \} = 1 - \frac{\alpha}{\beta t} \).

Hence, we have

\[ \frac{d\tilde{e}}{dF} \bigg|_{dF/dt=1} = \frac{\beta t ph(\tilde{x})}{\tilde{v}_{22} Ft} \left( t - 2 \frac{\alpha}{\beta} \right) \]

Therefore, we obtain a result:

if \( \frac{1}{2} t \cong \frac{\alpha}{\beta} \), then \( \frac{d\tilde{e}}{dF} \bigg|_{dF/dt=1} \cong 0 \),

since \( \tilde{v}_{22} < 0 \) from Mathematical Appendix 7.

Mathematical Appendix 12.

\[ \tilde{T} = t \{ ph(\tilde{x}) - w(\tilde{x}) \tilde{x} \} - t \tilde{e} ph(\tilde{x}) \{ 1 - q(\tilde{e}) F \} \]

In the same way as Mathematical Appendix 4, we have
\[
\frac{\partial \tilde{T}}{\partial F} = t \{ ph(\tilde{x}) - w(\tilde{x}) \tilde{x} - w(\tilde{x}) \} \frac{\partial \tilde{x}}{\partial F} \\
- t \varepsilon \frac{\partial \tilde{x}}{\partial F} \\
+ t \varepsilon ph(\tilde{x}) q(\tilde{\varepsilon}) \{ 1 - q(\tilde{\varepsilon}) F - \varepsilon q'(\tilde{\varepsilon}) F \} tph(\tilde{x}) \frac{\partial \tilde{\varepsilon}}{\partial F} > 0,
\]

since \( ph(\tilde{x}) - w'(\tilde{x}) \tilde{x} - w(\tilde{x}) < 0 \) from Mathematical Appendix 8, \( \partial \tilde{x} / \partial F < 0 \) from (29), \( 1 - q(\tilde{\varepsilon}) F > 0 \) from Mathematical Appendix 7, \( 1 - q(\tilde{\varepsilon}) F - \varepsilon q'(\tilde{\varepsilon}) F > 0 \) from Mathematical Appendix 9, and \( \partial \tilde{\varepsilon} / \partial F < 0 \) from (30).

Mathematical Appendix 13.

\[
\frac{\partial \tilde{T}}{\partial t} = \{ 1 - \tilde{\varepsilon} + \varepsilon q(\tilde{\varepsilon}) F \} ph(\tilde{x}) - w(\tilde{x}) \tilde{x} \\
+ t \{ ph(\tilde{x}) - w'(\tilde{x}) \tilde{x} - w(\tilde{x}) \} \frac{\partial \tilde{x}}{\partial t} \\
- t \varepsilon ph(\tilde{x}) \{ 1 - q(\tilde{\varepsilon}) F \} \frac{\partial \tilde{x}}{\partial t} \\
- t ph(\tilde{x}) \{ 1 - q(\tilde{\varepsilon}) F - \varepsilon q'(\tilde{\varepsilon}) F \} \frac{\partial \tilde{\varepsilon}}{\partial t}
\]

Except last term concerning \( \partial \tilde{\varepsilon} / \partial t \), this equation is similar to that of Mathematical Appendix 5. \( 1 - q(\tilde{\varepsilon}) F - \varepsilon q'(\tilde{\varepsilon}) F > 0 \) from Mathematical Appendix 8 and \( \partial \tilde{\varepsilon} / \partial t > 0 \) from (28), then the last term is negative. \( ph'(\tilde{x}) - w'(\tilde{x}) \tilde{x} - w(\tilde{x}) < 0 \) from Mathematical Appendix 8, \( \partial \tilde{x} / \partial t > 0 \) from (27), \( 1 - q(\tilde{\varepsilon}) F > 0 \) Mathematical Appendix 7, then other terms except the first term are negative. Therefore, the sign of \( \partial \tilde{T} / \partial t \) cannot be determined.

Mathematical Appendix 14.

Differentiating the first-order conditions (23) and (24) with respect to \( \alpha \) yields
\[
\begin{pmatrix}
\bar{v}_{11} & \bar{v}_{12} \\
\bar{v}_{21} & \bar{v}_{22}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \bar{v}}{\partial \bar{a}} \\
\frac{\partial \bar{\varepsilon}}{\partial \bar{a}}
\end{pmatrix} =
\begin{pmatrix}
-\bar{v}_{15} \\
-\bar{v}_{25}
\end{pmatrix},
\]
where \(\bar{v}_{15} = \frac{\partial^2 \bar{v}}{\partial \bar{x} \partial \bar{a}}\) and \(\bar{v}_{25} = \frac{\partial^2 \bar{v}}{\partial \bar{\varepsilon} \partial \bar{a}}\).

Hence, we get
\[
\frac{\partial \bar{x}}{\partial \bar{a}} = \frac{(-\bar{v}_{15} \bar{v}_{22} + \bar{v}_{25} \bar{v}_{12})}{D} > 0,
\]
since \(\bar{v}_{12} = 0\) and \(\bar{v}_{22} < 0\) from Mathematical Appendix 7, \(\bar{D} > 0\) from (26), and \(\bar{v}_{15} = (1-\bar{\varepsilon})p h'(\bar{x}) > 0\).

Mathematical Appendix 15.
\[
\frac{\partial \bar{\varepsilon}}{\partial \bar{a}} = \frac{(-\bar{v}_{11} \bar{v}_{25} + \bar{v}_{21} \bar{v}_{15})}{\bar{D}} < 0
\]
since \(\bar{v}_{21} = 0\) from Mathematical Appendix 7, \(\bar{D} > 0\) from (26), \(\bar{v}_{11} < 0\) from Mathematical Appendix 7, and \(\bar{v}_{25} = -p h'(\bar{x}) < 0\).

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