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Covering a bounded set of functions by an increasing chain of slaloms

Masaru Kada*

Abstract

A slalom is a sequence of finite sets of length $\omega$. Slaloms are ordered by coordinatewise inclusion with finitely many exceptions. Improving earlier results of Mildenberger, Shelah and Tsaban, we prove consistency results concerning existence and non-existence of an increasing sequence of a certain type of slaloms which covers a bounded set of functions in $\omega^\omega$.

1 Introduction

We use standard terminology and refer the readers to [2] for undefined set-theoretic notions.

Bartoszyński [1] introduced the combinatorial concept of slalom to study combinatorial aspects of measure and category on the real line.

We call a sequence of finite subsets of $\omega$ of length $\omega$ a slalom. For a function $g \in \omega^\omega$, let $S^g$ be the set of slaloms $\varphi$ such that $|\varphi(n)| \leq g(n)$ for all $n < \omega$. $S$ denotes $S^g$ for $g(n) = 2^n$. For two slaloms $\varphi$ and $\psi$, we write $\varphi \sqsubseteq \psi$ if $\varphi(n) \subseteq \psi(n)$ for all but finitely many $n < \omega$. For a function $f \in \omega^\omega$ and a slalom $\varphi$, $f \sqsubseteq \varphi$ if $\langle \{f(n)\} : n < \omega \rangle \subseteq \varphi$.

Mildenberger, Shelah and Tsaban [9] defined cardinals $\theta_h$ for $h \in \omega^\omega$ and $\theta_*$ to give a partial characterization of the cardinal $\mathfrak{o}$, the critical cardinality of a certain selection principle for open covers.

The definition of $\theta_h$ in [9] is described using a combinatorial property which is called o-diagonalization. Here we redefine $\theta_h$ to fit in the present

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context. It is easy to see that the following definition is equivalent to the original one. For a function \( h \in (\omega \setminus \{0, 1\})^\omega \), let \( h - 1 \) denote the function \( h' \in \omega^\omega \) which is defined by \( h'(n) = h(n) - 1 \) for all \( n \).

**Definition 1.1.** For a function \( h \in (\omega \setminus \{0, 1\})^\omega \), \( \theta_h \) is the smallest size of a subset \( \Phi \) of \( S^{h-1} \) which satisfies the following, if such a set \( \Phi \) exists:

1. \( \Phi \) is well-ordered by \( \sqsubseteq \);
2. For every \( f \in \prod_{n<\omega} h(n) \) there is a \( \varphi \in \Phi \) such that \( f \sqsubseteq \varphi \).

If there is no such \( \Phi \), we define \( \theta_h = c^+ \).

It is easy to see that \( h_1 \leq^* h_2 \) implies \( \theta_{h_1} \geq \theta_{h_2} \).

**Definition 1.2 ([9]).** \( \theta* = \min \{ \theta_h : h \in \omega^\omega \} \).

In Section 2, we will show that \( \theta* = c^+ \) is consistent with ZFC.

We say a proper forcing notion \( \mathbb{P} \) has the Laver property if, for any \( h \in \omega^\omega \), \( p \in \mathbb{P} \) and a \( \mathbb{P} \)-name \( \dot{f} \) for a function in \( \omega^\omega \) such that \( p \Vdash \dot{f} \in \prod_{n<\omega} h(n) \), there exist \( q \in \mathbb{P} \) and \( \varphi \in \mathcal{S} \) such that \( q \) is stronger than \( p \) and \( q \Vdash \dot{f} \sqsubseteq \varphi \).

Mildenberger, Shelah and Tsaban proved that \( \theta* = \aleph_1 \) holds in all forcing models by a proper forcing notion with the Laver property over a model for CH, the continuum hypothesis [9]. In section 2, we refine their result and state a sufficient condition for \( \theta* \leq c \). As a consequence, we will show that Martin’s axiom implies \( \theta* = c \).

In Section 3, we give an application of the lemma presented in Section 2 to another problem in topology. We answer a question on approximations to the Stone–Čech compactification of \( \omega \) by Higson compactifications of \( \omega \), which was posed by Kada, Tomoyasu and Yoshinobu [6].

## 2 Facts on the cardinal \( \theta* \)

First we observe that \( \theta* = c^+ \) is consistent with ZFC. We use the following theorem, which is a corollary of Kunen’s classical result [7]. For the readers’ convenience, we present a complete proof in Section 4.

**Theorem 2.1.** Suppose that \( \kappa \geq \aleph_2 \). The following holds in the forcing model obtained by adding \( \kappa \) Cohen reals over a model for CH: Let \( \mathcal{X} \) be a Polish space and \( A \subseteq \mathcal{X} \times \mathcal{X} \) a Borel set. Then there is no sequence \( \langle r_\alpha : \alpha < \omega_2 \rangle \) in \( \mathcal{X} \) which satisfies

\[
\alpha \leq \beta < \omega_2 \text{ if and only if } (r_\alpha, r_\beta) \in A.
\]
Fix $h \in \omega^\omega$. We may regard $S^{h^{-1}}$ as a product space of countably many finite discrete spaces, and then the relation $\sqsubseteq$ on $S^{h^{-1}}$ is a Borel subset of $S^{h^{-1}} \times S^{h^{-1}}$.

**Theorem 2.2.** $\theta_\ast = c^+$ holds in the forcing model obtained by adding $\aleph_2$ Cohen reals over a model for CH.

**Proof.** Fix $h \in \omega^\omega$. By Theorem 2.1, in the forcing model obtained by adding $\aleph_2$ Cohen reals over a model for CH, there is no $\sqsubseteq$-increasing chain of length $\omega_2$ in $S^{h^{-1}}$. This means that $\theta_h$ must be $\aleph_1$ whenever $\theta_h \leq c$.

On the other hand, $\text{cov}(\mathcal{M}) = \aleph_2$ holds in the same model. Also, by [9] we have $\text{cov}(\mathcal{M}) \leq \omega \delta \leq \theta_h$. This means that $\theta_h$ cannot be $\aleph_1$ in this model, and hence $\theta_h = c^+$. \hfill \Box

Next we state a sufficient condition for $\theta_\ast \leq c$. We use the following characterization of $\text{add}(\mathcal{N})$.

**Theorem 2.3 ([2, Theorem 2.3.9]).** $\text{add}(\mathcal{N})$ is the smallest size of a subset $F$ of $\omega^\omega$ such that, for every $\varphi \in S$ there is an $f \in F$ such that $f \nsubseteq \varphi$.

**Definition 2.4 ([5, Section 5]).** For a function $h \in \omega^\omega$, $l_h$ is the smallest size of a subset $\Phi$ of $S$ such that for all $f \in \prod_{n<\omega} h(n)$ there is a $\varphi \in \Phi$ such that $f \sqsubseteq \varphi$. Let $l = \sup \{l_h : h \in \omega^\omega\}$.

Note that $h_1 \leq^* h_2$ implies $l_{h_1} \leq l_{h_2}$.

If CH holds in a ground model $V$, $h \in \omega^\omega \cap V$, and a proper forcing notion $P$ has the Laver property, then $l_h = \aleph_1$ holds in the model $V^P$. Consequently, if CH holds in $V$, $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ is a countable support iteration of proper forcings, $P = \lim_{\alpha<\omega_2} P_\alpha$ and

$$\Vdash_{P_\alpha} "\text{the size of } Q_\alpha \text{ has the Laver property}"$$

holds for every $\alpha < \omega_2$, then $l = \aleph_1$ holds in $V^P$, since every function $h$ in $V^P$ appears in $V^{\check{\alpha}}$ for some $\alpha < \omega_2$, where CH holds.

Now we define a subset $S^+$ of $S$ as follows:

$$S^+ = \left\{ \varphi \in S : \lim_{n \to \infty} \frac{|\varphi(n)|}{2^n} = 0 \right\}.$$  

Let $l'_h$ be the smallest size of a subset $\Phi$ of $S^+$ such that for all $f \in \prod_{n<\omega} h(n)$ there is a $\varphi \in \Phi$ such that $f \sqsubseteq \varphi$. Clearly we have $l_h \leq l'_h$, and it is easy to see that for every $h \in \omega^\omega$ there is an $h^* \in \omega^\omega$ such that $l'_h \leq l_{h^*}$. Hence we have $l = \sup \{l'_h : h \in \omega^\omega\}$.

---

1In the paper [6], the authors state “If CH holds in a ground model $V$, and a proper forcing notion $P$ has the Laver property, then $l = \aleph_1$ holds in the model $V^{P\alpha}$. But it is inaccurate, since we do not see the values of $l_h$ for functions $h \in V^P$ which are not bounded by any function from $V$. 

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Lemma 2.5. For a subset $\Phi$ of $S^+$ of size less than $\text{add}(N)$, there is a $\psi \in S^+$ such that $\varphi \subseteq \psi$ for all $\varphi \in \Phi$.

Proof. For each $\varphi \in S^+$, define an increasing function $\eta_\varphi \in \omega^\omega$ by letting

$$
\eta_\varphi(m) = \min \left\{ l < \omega : \forall k \geq l \left( |\varphi(k)| < \frac{2^k}{m \cdot 2^m} \right) \right\}
$$

for all $m < \omega$. $\eta_\varphi$ is well-defined by the definition of $S^+$.

Suppose $\kappa < \text{add}(N)$ and fix a set $\Phi \subseteq S^+$ of size $\kappa$ arbitrarily. Since $\kappa < \text{add}(N) \leq \beta$, there is a function $\eta \in \omega^\omega$ such that $\lim_{n \to \omega} \eta(n)/2^n = \infty$ and for all $\varphi \in \Phi$ we have $\eta_\varphi \leq^* \eta$. For each $m < \omega$, let $I_m = \{ \eta(m), \eta(m) + 1, \ldots, \eta(m + 1) - 1 \}$ and enumerate $\prod_{n \in I_m} \omega^{\leq 2^n/(2^m - n)}$ as $\{s_{m,i} : i < \omega\}$, where $[r]$ denotes the largest integer which does not exceed the real number $r$.

For $\varphi \in \Phi$, define $\hat{\varphi} \in \omega^\omega$ as follows. If there is an $i < \omega$ such that $\varphi \upharpoonright I_m = s_{m,i}$, then let $\hat{\varphi}(m) = i$; otherwise $\hat{\varphi}(m)$ is arbitrary.

Since $|\Phi| = \kappa < \text{add}(N)$ and by Theorem 2.3, there is a $\psi \in S$ such that, for all $\varphi \in \Phi$ we have $\hat{\varphi} \subseteq \psi$. Define $\psi$ by letting for each $n$, if $n \in I_m$ then $\psi(n) = \bigcup \{s_{m,i} : i \in \hat{\varphi}(m)\}$, and if $n < \eta(0)$ then $\psi(n) = \emptyset$. It is straightforward to check that $\psi \in S^+$ and $\varphi \subseteq \psi$ for all $\varphi \in \Phi$.

Lemma 2.6. Suppose that $h \in \omega^\omega$ satisfies $h(n) > n^2$ for all $n < \omega$. If $\text{add}(N) = 1'_h$, then there is an $\subseteq$-increasing sequence $\langle \sigma_\alpha : \alpha < \kappa \rangle$ in $S^+$ such that, for all $f \in \prod_{n \in \omega} h(n)$ there is an $\alpha < \kappa$ such that $f \subseteq \varphi_\alpha$.

Proof. Fix a sequence $\langle \varphi_\alpha : \alpha < \kappa \rangle$ in $S^+$ so that for all $f \in \prod_{n \in \omega} h(n)$ there is an $\alpha < \kappa$ such that $f \subseteq \varphi_\alpha$. Using the previous lemma, inductively construct an $\subseteq$-increasing sequence $\langle \sigma_\alpha : \alpha < \kappa \rangle$ of elements of $S^+$ so that $\varphi_\alpha \subseteq \sigma_\alpha$ holds for each $\alpha < \omega_2$. Then $\langle \sigma_\alpha : \alpha < \kappa \rangle$ is as required.

Define $H_1 \in \omega^\omega$ by letting $H_1(n) = 2^n + 1$ for all $n$.

Theorem 2.7. If $\text{add}(N) = 1'_H$, then $\theta_* = \omega \cdot \text{add}(\mathcal{M})$.

Proof. Let $\kappa = \text{add}(N) = 1'_H$. Since $S^+ \subseteq S \subseteq S^{H_1-1}$, the previous lemma shows that $\theta_* \leq \theta_{H_1} \leq \kappa$. On the other hand, by [9], we have $\kappa = \text{add}(N) \leq \text{cov}(\mathcal{M}) \leq \omega \cdot \text{add}(\mathcal{M}) \leq \theta_*$.  

Corollary 2.8 ([9]). If a ground model $V$ satisfies CH, and a proper forcing notion $\mathbb{P}$ has the Laver property, then $\theta_* = \aleph_1$ holds in the model $V^{\mathbb{P}}$.

Proof. Follows from Theorem 2.7 and the fact that $\text{add}(N) = 1'_H = \text{le} = \aleph_1$ holds in the model $V^{\mathbb{P}}$.

Corollary 2.9. Martin’s axiom implies $\theta_* = \mathfrak{c}$.

Proof. Follows from Theorem 2.7 and the fact that $\text{add}(N) = 1'_H = 1 = \mathfrak{c}$ holds under Martin’s axiom.


3 Application

In this section, we give an answer to a question which was posed by Kada, Tomoyasu and Yoshinobu [6]. We refer the reader to [6] for undefined topological notions.

For compactifications $\alpha X$ and $\gamma X$ of a completely regular Hausdorff space $X$, we write $\alpha X \leq \gamma X$ if there is a continuous surjection from $\gamma X$ to $\alpha X$ which fixes the points from $X$, and $\alpha X \simeq \gamma X$ if $\alpha X \leq \gamma X \leq \alpha X$. The Stone–Čech compactification $\beta X$ of $X$ is the maximal compactification of $X$ in the sense of the order relation $\leq$ among compactifications of $X$.

For a proper metric space $(X, d)$, $X^d$ denotes the Higson compactification of $X$ with respect to the metric $d$.

ht is the smallest size of a set $D$ of proper metrics on $\omega$ such that

1. $\{\omega^d : d \in D\}$ is well-ordered by $\leq$;
2. There is no $d \in D$ such that $\omega^d \simeq \beta \omega$;
3. $\beta \omega \simeq \sup\{\omega^d : d \in D\}$, where sup is in the sense of the order relation $\leq$ among compactifications of $\omega$;

if such a set $D$ exists. We define $ht = c^+$ if there is no such $D$.

Kada, Tomoyasu and Yoshinobu [6, Theorem 6.16] proved the consistency of $ht = c^+$ using a similar argument to the proof of Theorem 2.2. But the consistency of $ht \leq c$ was not addressed. Here we state a sufficient condition for $ht \leq c$, and show that it is consistent with ZFC.

Define $H_2 \in \omega^\omega$ by letting $H_2(n) = 2^{2^{(n+4)}}$ for all $n$. The following lemma is obtained as a corollary of the proof of [6, Theorem 6.11].

Lemma 3.1. Let $\kappa$ be a cardinal. If there is an $\sqsubseteq$-increasing sequence $\langle \varphi_\alpha : \alpha < \kappa \rangle$ of slaloms in $S$ such that for all $f \in \prod_{n<\omega} H_2(n)$ there is an $\alpha < \kappa$ such that $f \sqsubseteq \varphi_\alpha$, then $ht \leq \kappa$.

Now we have the following theorem.

Theorem 3.2. If $\text{add}(\mathcal{N}) = \text{l}_{H_2}$, then $ht = \text{add}(\mathcal{N})$.

Proof. $\text{add}(\mathcal{N}) \leq ht$ is proved in [6, Section 6]. To see $ht \leq \text{add}(\mathcal{N})$, apply Lemma 2.6 for $h = H_2$ to get a sequence of slaloms which is required in Lemma 3.1. \qed

Corollary 3.3. If a ground model $V$ satisfies CH, and a proper forcing notion $\mathbb{P}$ has the Laver property, then $ht = \aleph_1$ holds in the model $V^\mathbb{P}$.
Proof. Follows from Theorem 3.2 and the fact that $\text{add}(\mathcal{N}) = \mathcal{I}_{H_2} = \mathcal{I} = \aleph_1$ holds in the model $V^\mathcal{P}$.

**Corollary 3.4.** Martin’s axiom implies $\mathfrak{ht} = \mathfrak{c}$.

Proof. Follows from Theorem 3.2 and the fact that $\text{add}(\mathcal{N}) = \mathcal{I}_{H_2} = \mathcal{I} = \mathfrak{c}$ holds under Martin’s axiom.

### 4 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. The idea of the proof is the same as the one in Kunen’s original proof [7], which is known as the “isomorphism of names” argument. The same argument is also found in [4].

For an infinite set $I$, let $\mathbb{C}(I) = \text{Fn}(I, 2, \aleph_0)$, the canonical Cohen forcing notion for the index set $I$. As described in [8, Chapter 7], for any $\mathbb{C}(I)$-name $\dot{r}$ for a subset of $\omega$, we can find a countable subset $J$ of $I$ and a nice $\mathbb{C}(J)$-name $\dot{s}$ for a subset of $\omega$ such that $\Vdash_{\mathbb{C}(I)} \dot{s} = \dot{r}$. For a countable set $I$, there are only $\mathfrak{c}$ nice $\mathbb{C}(I)$-names for subsets of $\omega$.

**Proof of Theorem 2.1.** Suppose that $\kappa \geq \aleph_2$. Let $X$ be a Polish space, $\dot{A}$ a $\mathbb{C}(\kappa)$-name for a Borel subset of $X \times X$, and $\langle \dot{r}_\alpha : \alpha < \omega_2 \rangle$ a sequence of $\mathbb{C}(\kappa)$-names for elements of $X$.

We will prove the following statement:

$$\Vdash_{\mathbb{C}(\kappa)} \exists \alpha < \omega_2 \exists \beta < \omega_2 (\alpha < \beta \land (\langle \dot{r}_\alpha, \dot{r}_\beta \rangle \notin \dot{A} \lor \langle \dot{r}_\beta, \dot{r}_\alpha \rangle \in \dot{A})).$$

There is nothing to do if it holds that

$$\Vdash_{\mathbb{C}(\kappa)} \exists \alpha < \omega_2 \exists \beta < \omega_2 (\alpha < \beta \land (\langle \dot{r}_\alpha, \dot{r}_\beta \rangle \notin \dot{A})).$$

So we assume that it fails, and fix any $p \in \mathbb{C}(\kappa)$ which satisfies

$$p \Vdash_{\mathbb{C}(\kappa)} \forall \alpha < \omega_2 \forall \beta < \omega_2 (\alpha < \beta \rightarrow (\langle \dot{r}_\alpha, \dot{r}_\beta \rangle \in \dot{A}).$$

We will find $\alpha, \beta < \omega_2$ such that $\alpha < \beta$ and $p \Vdash_{\mathbb{C}(\kappa)} \langle \dot{r}_\beta, \dot{r}_\alpha \rangle \in \dot{A}$, which concludes the proof.

Let $J_p = \text{dom}(p)$. Find a set $J_\alpha \in [\kappa]^{\aleph_0}$ and a nice $\mathbb{C}(J_\alpha)$-name $\dot{C}_\alpha$ for a subset of $\omega$ such that

$$\Vdash_{\mathbb{C}(\kappa)} \text{"} \dot{C}_\alpha \text{ is a Borel code of } \dot{A}. \text{"}$$

For each $\alpha < \omega_2$, find a set $J_\alpha \in [\kappa]^{\aleph_0}$ and a nice $\mathbb{C}(J_\alpha)$-name $\dot{C}_\alpha$ for a subset of $\omega$ such that

$$\Vdash_{\mathbb{C}(\kappa)} \text{"} \dot{C}_\alpha \text{ is a Borel code of } \{\dot{r}_\alpha\}. \text{"}$$
Using the $\Delta$-system lemma [8, II Theorem 1.6], take $S \in [\kappa]^\aleph_0$ and $K \subseteq [\omega_2]^{\aleph_2}$ so that $J_p \cup J_A \cup (J_\alpha \cap J_\beta) \subseteq S$ for any $\alpha, \beta \in K$ with $\alpha \neq \beta$. Without loss of generality we may assume that $|J_\alpha \cap S| = \aleph_0$ for all $\alpha \in K$. For each $\alpha \in K$, enumerate $J_\alpha \setminus S$ as $\langle \delta^\alpha_n : n < \omega \rangle$.

For $\alpha, \beta \in K$, and let $\sigma_{\alpha, \beta}$ be the involution (automorphism of order 2) of $\mathbb{C}(\kappa)$ obtained by the permutation of coordinates which interchanges $\delta^\alpha_n$ with $\delta^\beta_n$ for each $n$. $\sigma_{\alpha, \beta}$ naturally induces an involution of the class of all $\mathbb{C}(\kappa)$-names: We simply denote it by $\sigma_{\alpha, \beta}$. Since $J_p \cup J_A \subseteq S$, for all $\alpha, \beta \in K$ we have $\sigma_{\alpha, \beta}(p) = p$, $\sigma_{\alpha, \beta}(C_A) = \hat{C}_A$ and $\Vdash_{\mathbb{C}(\kappa)} \sigma_{\alpha, \beta}(\check{A}) = \check{A}$.

Since $|K| = \aleph_2$ and there are only $\mathfrak{c} = \aleph_1$ nice names for subsets of $\omega$ over a countable index set, we can find $\alpha, \beta \in K$ with $\alpha < \beta$ such that $\sigma_{\alpha, \beta}(\hat{C}_\alpha) = \hat{C}_\beta$. Then $\sigma_{\alpha, \beta}(\hat{C}_\beta) = \hat{C}_\alpha$ and

$$\Vdash_{\mathbb{C}(\kappa)} \" \sigma_{\alpha, \beta}(\check{r}_\alpha) = \check{r}_\beta \text{ and } \sigma_{\alpha, \beta}(\check{r}_\beta) = \check{r}_\alpha. \"$$

By (*), we have $p \Vdash_{\mathbb{C}(\kappa)} \langle \check{r}_\alpha, \check{r}_\beta \rangle \in \check{A}$. Since $\sigma_{\alpha, \beta}$ is an automorphism of $\mathbb{C}(\kappa)$, we have

$$\sigma_{\alpha, \beta}(p) \Vdash_{\mathbb{C}(\kappa)} \langle \sigma_{\alpha, \beta}(\check{r}_\alpha), \sigma_{\alpha, \beta}(\check{r}_\beta) \rangle \in \sigma_{\alpha, \beta}(\check{A})$$

and hence $p \Vdash_{\mathbb{C}(\kappa)} \langle \check{r}_\beta, \check{r}_\alpha \rangle \in \check{A}$. \hfill \qed

**Remark 1.** Fuchino pointed out that Theorem 2.1 is generalized in the following two ways [3]: (1) The set $A$ is not necessarily Borel, but is “definable” by some formula. (2) We can prove a similar result for a forcing extension by a side-by-side product of the same forcing notions, each generically adds a real in a natural way. The argument in the above proof also works in those generalized settings.

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**References**


